Higher-Order Relational Refinement Types for Mechanism Design and Differential Privacy

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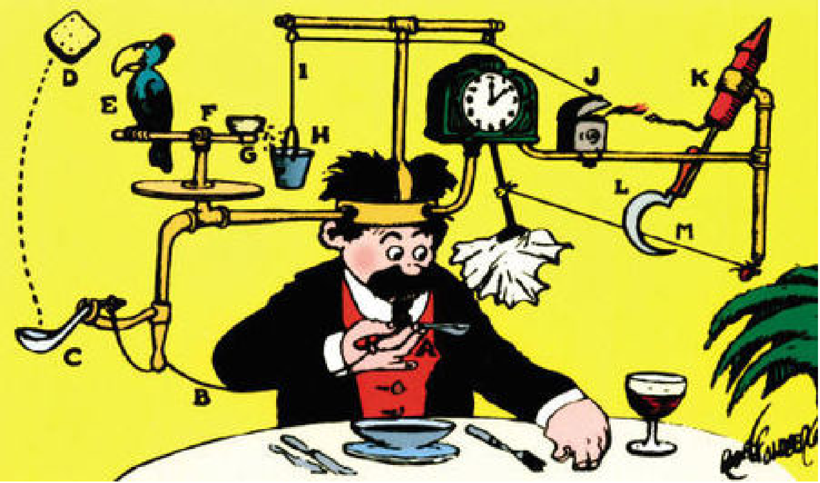
January 15th, 2015

The Application



Mechanism Design

One painting for sale

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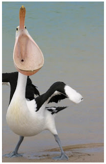
How much will you pay?



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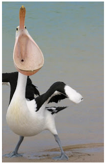

$50 million!

$3

A story

One painting for sale

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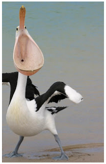

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Who wins, and for how much?

How much will you pay?

Top bid pays top price?

- Simple rule
- Can encourage manipulation...

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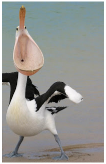

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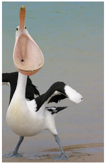

~~$50 million!~~
$10.1 million?

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Algorithm design with strategic inputs

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Rational agents

- Report data
- Care about output
- May lie, strategize

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Goal: encourage "good" behavior

Designing auctions

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$\boxed{\text{Property: agent always maximizes happiness with } b = v}$

A (very) simple auction

Fixed price auction

- Given a fixed price `price`
- Bidder bids `bid`, buys item if higher than `price`

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What is the happiness function for a bidder?

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fixedprice price value bid =
  if bid > price then
    value - price
  else
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```

Consider bidder's happiness function…

- First run: bidder bids $b = v$ (honest)

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The verification strategy

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This is a relational property

Introducing HOARe^2

A type system with relational refinement types

Judgment

$$\Gamma \vdash e : \{x : \ T \ \mid \ \phi(x) \ \}$$

Judgment

$$\Gamma \vdash e : \{x : \overset{\text{type}}{\boxed{T}} \mid \phi(x)\}$$

Judgment

type

predicate

$$\Gamma \vdash e : \{x : \boxed{T} \mid \boxed{\phi(x)}\}$$

Judgment

type

predicate

$$\Gamma \vdash e : \{x : T \mid \phi(x)\}$$

"e is a program of type T such that $\phi(e)$ holds"

Example

$$\Gamma \vdash 3 : \{x : \mathbb{Z} \mid x \geqslant 0\}$$

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"3 is a non-negative integer"

Relational Judgment

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$$\{y :: \mathbb{Z} \mid y_{\lhd} \leqslant y_{\rhd}\} \vdash e :: \{x :: \mathbb{Z} \mid x_{\lhd} \leqslant x_{\rhd}\}$$

Relational Judgment

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$$\{y :: \mathbb{Z} \mid y_{\triangleleft} \leqslant y_{\triangleright}\} \vdash e :: \{x :: \mathbb{Z} \mid x_{\triangleleft} \leqslant x_{\triangleright}\}$$

"If y increases, then e increases."

Relational Judgment

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$$\{y :: \mathbb{Z} \mid y_\triangleleft \leqslant y_\triangleright\} \vdash e :: \{x :: \mathbb{Z} \mid x_\triangleleft \leqslant x_\triangleright\}$$

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Background

- First used in the RF* language, POPL 2014

Happiness function

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Typing truthfulness

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$$\rightarrow \{v :: \mathbb{R} \mid v_{\lhd} = v_{\rhd}\} \qquad \text{(Bidder value fixed)}$$

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&\to \{u :: \mathbb{R} \mid u_{\lhd} \geqslant u_{\rhd}\} && \text{(Truthful)}
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- Unlimited supply of items (e.g., music files)
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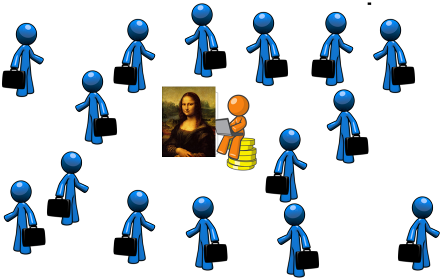

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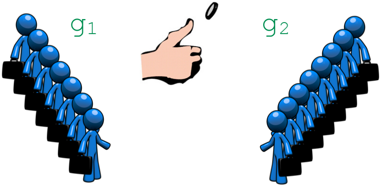

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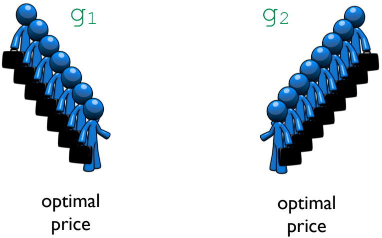

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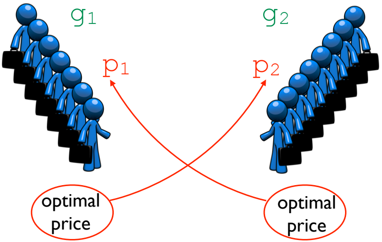

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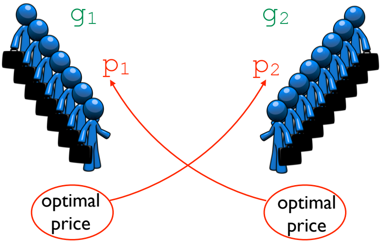

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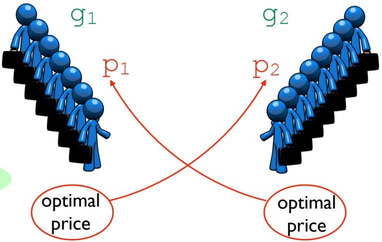

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One key ingredient

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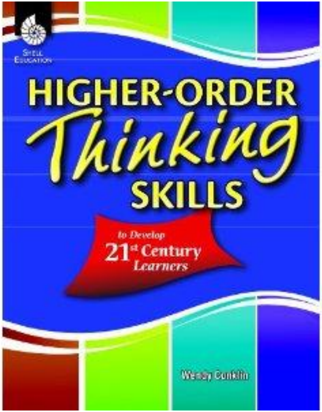
f_1 bigger than
f_2 on average

Extending HOARe^2



Distributions and Higher-order refinements

Relating Distributions

Probabilistic programs

- Reason about two runs of a probabilistic program
- Use type of probability distributions

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???

Equivalence of Distributions

$$\Gamma \vdash e :: \mathfrak{M}_{0,0}[\{x :: T \mid \phi(x_\triangleleft, x_\triangleright)\}]$$

What does this mean?

- Convert relation ϕ to a relation $\phi^\#$ on distributions over T
- Two runs of e related by $\phi^\#$ (as distributions!)

Example

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- Generalizing $0, 0$ to ε, δ models differential privacy

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Our contribution

- Simplify and build into a type system

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Our contribution

- Consistency by carefully handling termination
- Show naïve treatment leads to inconsistency

Want to show

$$\mathbb{E}\ \mu\ f_1 \geqslant \mathbb{E}\ \mu\ f_2$$

In HOARe^2, type $\mathbb{E}$ as...

$$\mathfrak{M}_{0,0}[\{x :: A \mid x_\lhd = x_\rhd\}] \qquad \text{(Same distributions)}$$

Expressing monotonicity of expectations

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\end{aligned}$$

Much more in the paper

Semantics

- Soundness of the system
- Requires termination

Implementation

- Automated, low annotation burden
- Why3 and SMT solvers

Translation

- Embedding of DFuzz, a language for differential privacy

More complex examples

- Verify differential privacy
- Verify MD properties beyond truthfulness

Takeaway points

Four features, one system

- HOARe^2: relational properties for randomized programs
- Combine features in a clean, usable way

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Formal verification for mechanism design!

- Exciting, under-explored area for verification
- Tons of interesting properties, mechanisms
- Strong motivation besides (mere) correctness

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