Computer-aided Verification in Mechanism Design

Gilles Barthe, Marco Gaboardi, Emilio Jesús Gallego Arias,
Justin Hsu*, Aaron Roth*, Pierre-Yves Strub

IMDEA Software, École Polytechnique,
University at Buffalo, *University of Pennsylvania

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Mechanism design

$=$

Algorithm design

$+$

Strategic inputs

*In computer science

Incentive properties

Encourage agents to behave simply

Benefits

- For the agents: easy to decide what to do
- For the designer: easy to reason about what agents will do

Best case: truthfulness

Model

- Agents have private type $t_i \in T$
- Mechanism inputs: agents report $s_i \in T$
- Mechanism outputs: outcome $o \in O$ and payments $p_i \in \mathbb{R}$

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Model

- Agents have private type $t_i \in T$
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Definition (Complete information)

A mechanism is truthful (DSIC) if each agent maximizes their utility by reporting $s_i = t_i$, no matter what other agents do.

Definition (Incomplete information)

A mechanism is Bayesian Incentive Compatible (BIC) if each agent maximizes their expected utility by reporting $s_i = t_i$, when other agents report their true type drawn from a known prior μ.

Mechanism $\approx$ Program

Truthfulness $\approx$ Property

Program verification
for incentive properties

But isn't this really hard?

Alan designed the perfect computer

Divide the task

- Proof construction: hard
- Proof checking: easy

Why verify properties? Check correctness

```python
import random

class GSP:
    """
    Implements the generalized second price auction mechanism.
    """
    @staticmethod
    def compute(slot_clicks, reserve, bids):
        """
        Given info about the setting (clicks for each slot, and reserve price),
        and bids (list of (id, bid) tuples), compute the following:
          allocation:  list of the occupant in each slot
              len(allocation) = min(len(bids), len(slot_clicks))
          per_click_payments: list of payments for each slot
              len(per_click_payments) = len(allocation)

        If any bids are below the reserve price, they are ignored.

        Returns a pair of lists (allocation, per_click_payments):
         - allocation is a list of the ids of the bidders in each slot
            (in order)
         - per_click_payments is the corresponding payments.
        """
        valid = lambda (a, bid): bid >= reserve
        valid_bids = filter(valid, bids)

        rev_cmp_bids = lambda (a1, b1), (a2, b2): cmp(b2, b1)
        # shuffle first to make sure we don't have any bias for lower or
        # higher ids
        random.shuffle(valid_bids)
        valid_bids.sort(rev_cmp_bids)

        num_slots = len(slot_clicks)
        allocated_bids = valid_bids[:num_slots]
        if len(allocated_bids) == 0:
            return ([], [])

        (allocation, just_bids) = zip(*allocated_bids)

        # Each pays the bid below them, or the reserve
        per_click_payments = list(just_bids[1:])  # first num_slots - 1 slots
        # figure out whether the last slot payment is set by the reserve or
        # the first non-allocated bidder
        if len(valid_bids) > num_slots:
            last_payment = valid_bids[num_slots][1]
        else:
            last_payment = reserve
        per_click_payments.append(last_payment)
        return (list(allocation), per_click_payments)

    @staticmethod
    def bid_range_for_slot(slot, slot_clicks, reserve, bids):
        """
        Compute the range of bids that would result in the bidder ending up
        in slot, given that the other bidders submit bidders.
```

$$
\begin{aligned}
\mathbf{E}[\bar{A}(t_1) \,\|\, w] &= \mathbf{E}[\bar{A}(t_1) \,\|\, w, \beta = k] \cdot \Pr[\beta = k \,\|\, w] \\
&\quad + \mathbf{E}[\bar{A}(t_1) \,\|\, w, \beta = \ell] \cdot \Pr[\beta = \ell \,\|\, w] \\
&= y_\ell \Pr[\beta = k \,\|\, w] + y_k \Pr[\beta = \ell \,\|\, w]
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}[\bar{A}(t_2) \,\|\, w] &= \mathbf{E}[\bar{A}(t_2) \,\|\, w, \beta = k] \cdot \Pr[\beta = k \,\|\, w] \\
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\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{E}[\bar{A}(t_2) - \bar{A}(t_1) \,\|\, w] \\
&\quad = (y_k - y_\ell) \cdot (\Pr[\beta = k \,\|\, w] - \Pr[\beta = \ell \,\|\, w]) \\
&= \frac{y_k - y_\ell}{\Pr(W = w)} \cdot \left(\Pr\left[\begin{array}{l} \beta = k, \\ W = w \end{array} \right] - \Pr\left[\begin{array}{l} \beta = \ell, \\ W = w \end{array} \right] \right).
\end{aligned}
$$

Every factor in the last line is non-negative, except possibly the probability difference $\Pr[\beta = k, W = w] - \Pr[\beta = \ell, W = w]$. To prove that this difference is in fact positive, we will in fact prove that

$$\Pr[\beta = k, W = w \,\|\, a, b, \mathbf{s}] \geq \Pr[\beta = \ell, W = w \,\|\, a, b, \mathbf{s}].$$

for all possible values of the random variables a, b, and $\mathbf{s}$. Note that when we condition on $a, b, \mathbf{s}$, the values of β, W determine the value of the vector $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \ldots, \widetilde{X}_m)$ and vice-versa. Specifically, since $\widetilde{\mathbf{X}}$ is the vector obtained from $\mathbf{s}/L$ by rearranging its entries using σ^{-1}, W constrains $\widetilde{\mathbf{X}}$ to be one of two possible vectors $\mathbf{z}, \mathbf{z}'$ that differ by interchanging their k^{th} and ℓ^{th} components. Assume without loss of generality that $z_k \geq z_\ell$. (Otherwise, simply rename $\mathbf{z}$ to $\mathbf{z}'$ and vice-versa.) Then

$$
\begin{aligned}
\Pr[\beta = k, W = w \,\|\, a, b, \mathbf{s}] &= \Pr[\widetilde{\mathbf{X}} = \mathbf{z} \,\|\, a, b, \mathbf{s}] \\
\text{(A.1)} \qquad &= \prod_{j=1}^{m} \Pr[L \cdot \widetilde{X}_j = z_j] \\
\Pr[\beta = \ell, W = w \,\|\, a, b, \mathbf{s}] &= \Pr[V \widetilde{X} = \mathbf{z}' \,\|\, a, b, \mathbf{s}]
\end{aligned}
$$

A.2 Proof of Lemma 4.13. In this section we restate and prove Lemma 4.13.

LEMMA A.1. *For any real numbers a, b, z in the interval $(0, 1)$ such that $a \leq b/2$, if q is any integer greater than $1/a$, then*

$$\min\{a, 1 - z\} < b(1 - z^q).$$

Proof. As $q > 1/a$ we have

$$(1 - a)^q < e^{-aq} < e^{-1} < \frac{1}{2}.$$

The proof consists of applying this inequality in two cases.

Case 1: $a < 1 - z$. In this case we have $z < 1 - a$, hence

$$1 - z^q > 1 - (1 - a)^q > \frac{1}{2}$$

$$\min\{a, 1 - z\} = a \leq \frac{b}{2} < b(1 - z^q),$$

as claimed.

Case 2: $a \geq 1 - z$. The equation

$$\frac{1 - z^q}{1 - z} = 1 + z + \cdots + z^{q-1}$$

reveals that $\frac{1-z^q}{1-z}$ is an increasing function of $z \in (0, 1)$. As $z \geq 1 - a$, we may conclude that

$$
\begin{aligned}
b\left(\frac{1 - z^q}{1 - z}\right) &\geq b\left(\frac{1 - (1 - a)^q}{1 - (1 - a)}\right) \\
&= \frac{b}{a}(1 - (1 - a)^q) > \frac{b}{a} \cdot \frac{1}{2} \geq 1,
\end{aligned}
$$

whence $b(1 - z^q) \geq 1 - z = \min\{a, 1 - z\}$, as desired.

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What if agents don't believe incentive property?

- Incentive properties often not obvious
- Read the proof (?)

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A possible model

- Designer constructs formal proof of incentive property
- Agents check it automatically

Our work: A case study

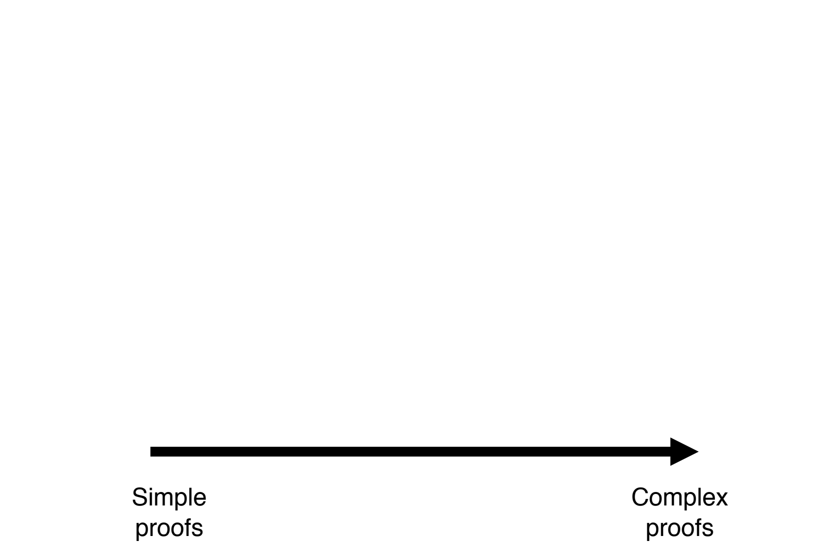
Target

- Replica-surrogate-matching mechanism (HKM)
- To prove: BIC

Proof is non-trivial

- Lots of reasoning about randomization
- Need incentive property for VCG mechanism

Proof construction approaches: basic tradeoff



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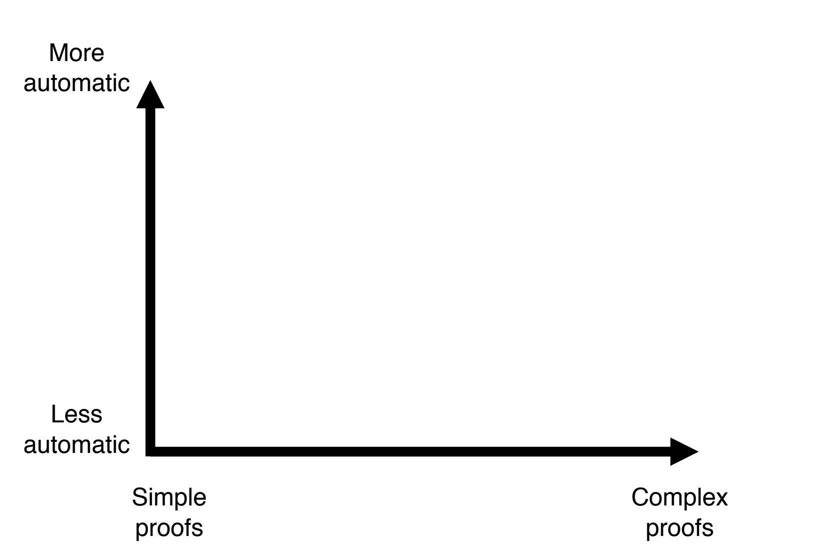

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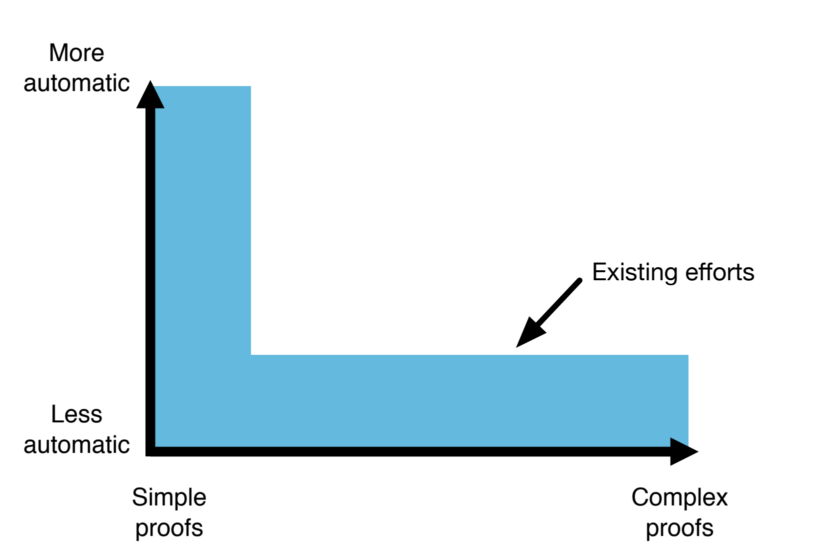

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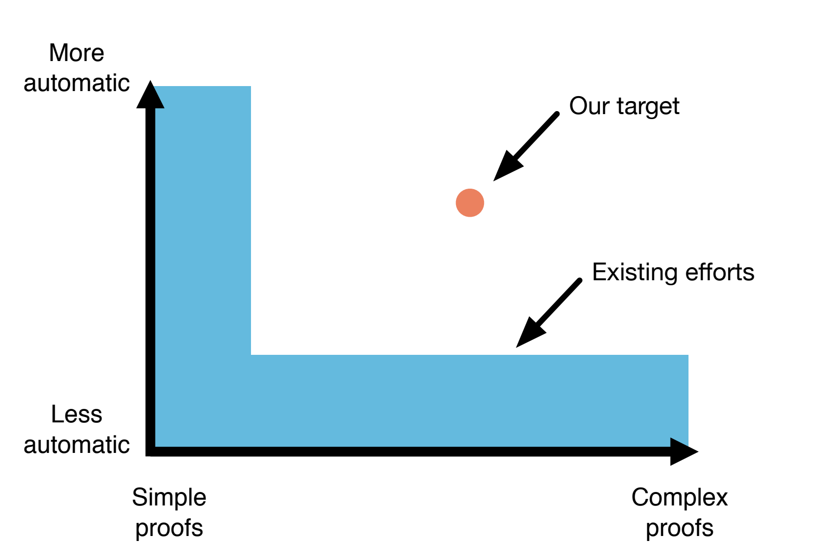

Idea: incentive properties are relational properties

Program: agent's report $\rightarrow$ agent's (expected) utility

- First run: agent report equal to agent type (truthful)
- Second run: agent report arbitrary (non-truthful)
- Truthfulness: first utility larger than second utility

Leverage specialized tools

- HOARe^2: for probabilistic relational properties

Formally verifying BIC

Four main steps

1. Write program
2. Annotate program with assertions
3. Apply solvers to automatically check assertions
4. Fall back to less automated approaches for remaining steps

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Writing the assertions

Basic form

$$\{prog :: S \mid \Phi(prog_1, prog_2)\}$$

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Incentive Compatibility

$$\{rept :: T \mid rept_1 = type\} \to \{util :: \mathbb{R} \mid util_1 \geq util_2\}$$

Applying solvers

Given $x_1 < x_2$, prove:

- $x_1 + 1 < x_2 + 2$ (easy)
- $f(x_1) < f(x_2)$, where f is a program (harder)

Applying solvers

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Results

- Almost all assertions (~ 60) automatically proved ($\sim$ seconds)
- Solvers run out of time on three assertions

Formally verifying BIC

Four main steps

1. Write program
2. Annotate program with assertions
3. Apply solvers to automatically check assertions
4. Fall back to less automated approaches for remaining steps

See paper for details!

Perspective

Promising signs: automatic parts

- Handle complex proofs and mechanisms
- Solvers usually work, and are fast

Pain points: manual parts

- When solvers fail: life is hard
- Crafting program and assertions

Needed: more case studies!

Do you have a mechanism that ...

- has a tedious proof?
- uses randomization?
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We want to know!

For brave souls: `https://github.com/ejgallego/HOARe2`

Needed: more case studies!

Do you have a mechanism that ...

- has a tedious proof?
- uses randomization?
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(Also, I am looking for a job ...)

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Writing the program

Main program: one agent's utility

- Input: agent's true type and report
- Output: agent's expected utility from mechanism
- Assume: other agents reports drawn from prior (BIC)

Top level code

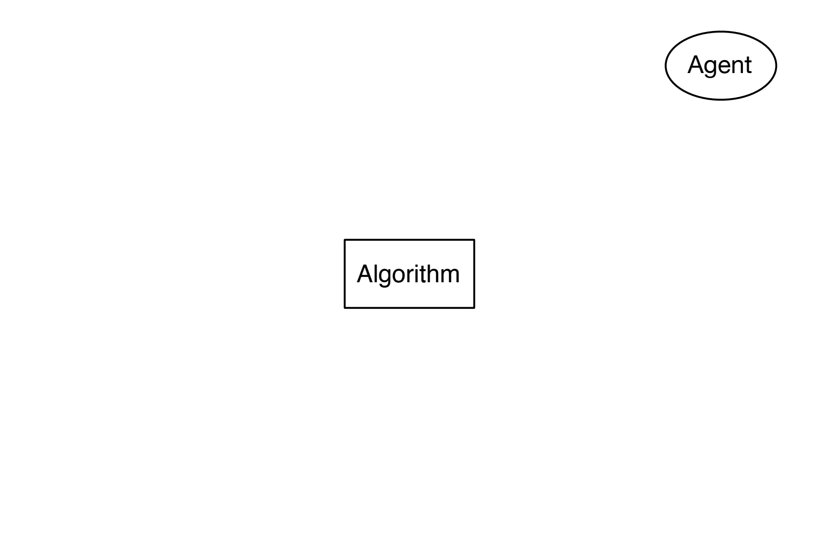
```
def Util (othermoves, myty, mybid) =
  𝔼 (c ∼ rsmcoins) {
    (mysur, mypay) = RStrans(c, myty, mybid);
    myval = 𝔼 {
      for i = 1 . . . n − 1:
        sample othersurs[i] ∼ (sample otherty ∼ mu; othermoves[i](otherty));
      algInput = (mysur, othersurs);
      outcome  = alg(algInput);
      return value (myty, outcome)
    };
    return (myval - mypay);
  }
```

Handling the hard assertions

Hardest step

- Mechanism transforms each report into a "surrogate" report
- Key lemma: if report $\sim$ prior, transformation preserves prior
- Manually construct proof in different system (EasyCrypt), $\sim$ 190 out of $\sim$ 260 total lines of manual proof

RSM mechanism (Hartline, Kleinberg, Malekian)



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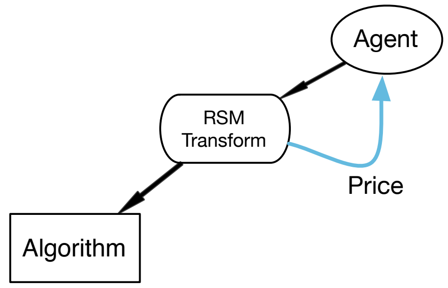

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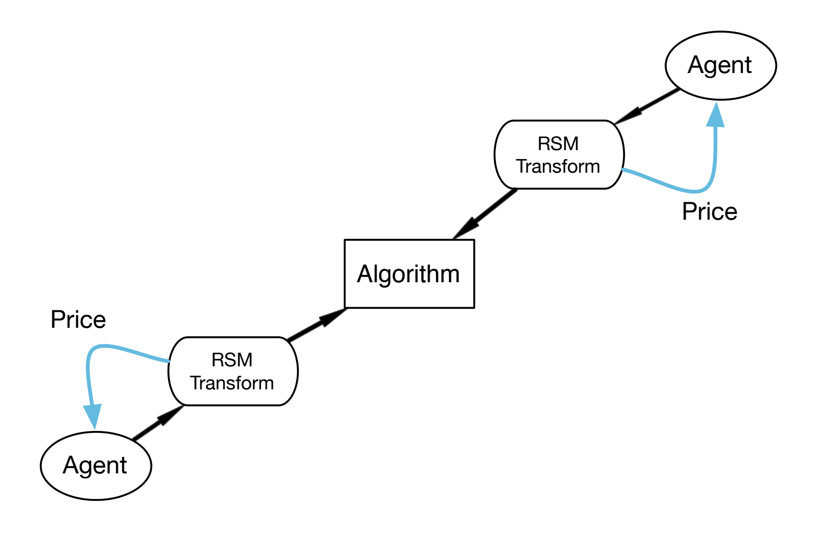

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