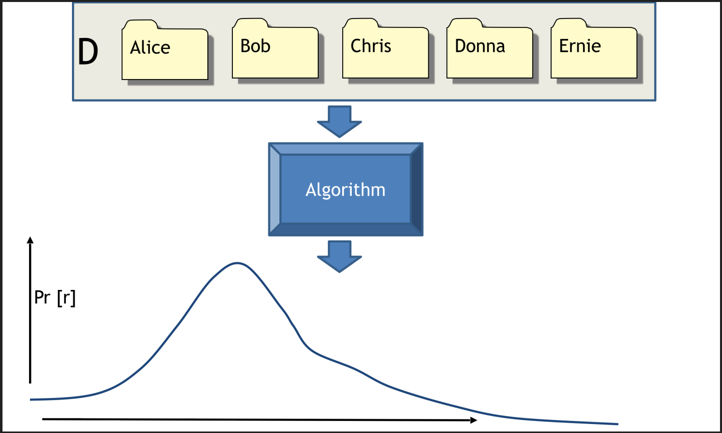
$\star$-Liftings for Differential Privacy and f-Divergences

Gilles Barthe, Thomas Espitau,
Justin Hsu, Tetsuya Sato, Pierre-Yves Strub

Differential privacy: probabilistic program property



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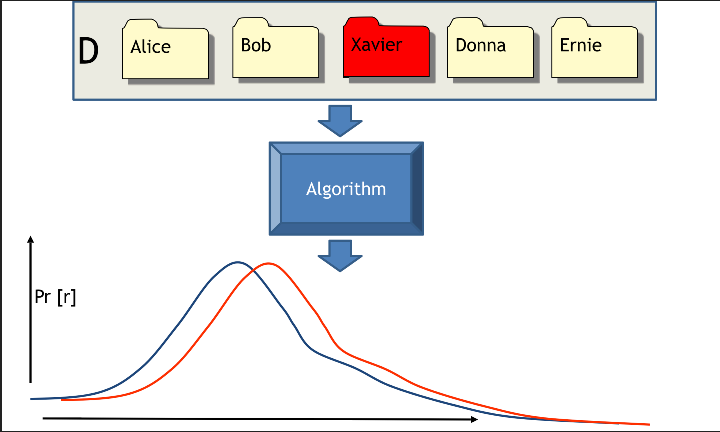

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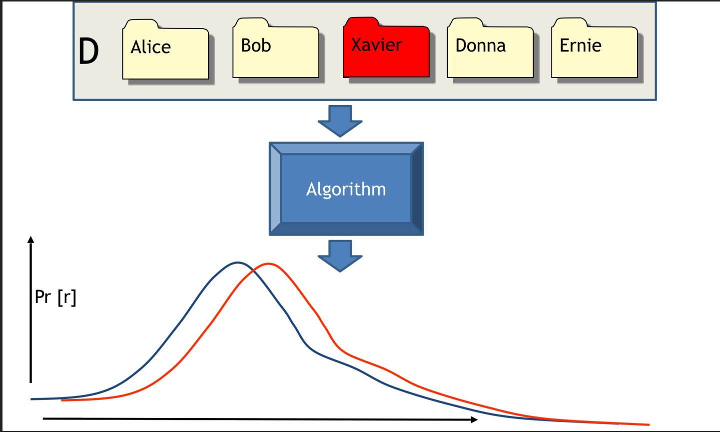

Output depends only a little
on any single individual's data

More formally

Definition (Dwork, McSherry, Nissim, Smith)
An algorithm is (ϵ, δ)-**differentially private** if, for every two adjacent inputs, the output distributions μ_1, μ_2 satisfy:

$$\boxed{\Delta_\epsilon(\mu_1, \mu_2) \leq \delta \triangleq \text{for all sets } S,\ \mu_1(S) \leq e^\epsilon \cdot \mu_2(S) + \delta}$$

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Behaves well under composition: "ϵ and δ add up"

Sequentially composing an (ϵ, δ)-private program and an (ϵ', δ')-private program is $(\epsilon + \epsilon', \delta + \delta')$-private.

How to verify this property?

Use ideas from probabilistic bisimulation

- $\Delta_\epsilon(\mu_1, \mu_2) \leq \delta$ means "approximately similar"
- Composition $\iff$ approximate probabilistic bisimulation

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Foundation for many styles of program verification

- Linear and dependent type systems
- Product program constructions
- Relational program logics

Review: Probabilistic Liftings and Approximate Liftings

Probabilistic liftings

Lift a binary relation R on pairs $S \times T$
to a relation $\langle R \rangle$ on distributions $\mathsf{Distr}(S) \times \mathsf{Distr}(T)$

Definition (Larsen and Skou)

Let $R \subseteq S \times T$ be a relation. Two distributions are related $\mu_1 \ \langle R \rangle \ \mu_2$ if there exists a witness $\eta \in \mathsf{Distr}(S \times T)$ such that:

1. $\pi_1(\eta) = \mu_1$ and $\pi_2(\eta) = \mu_2$,
2. $\eta(s, t) > 0$ only when $(s, t) \in R$.

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Example

$\mu_1 \; \langle = \rangle \; \mu_2$ is equivalent to $\mu_1 = \mu_2$.

An equivalent definition via Strassen's theorem

Theorem (Strassen 1965)
Let $R \subseteq S \times T$ be a relation. Then $\mu_1 \langle R \rangle \mu_2$ if and only if:

$$\boxed{\textit{for all subsets } A \subseteq S,\ \mu_1(A) \leq \mu_2(R(A))}$$

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Approximate liftings

Intuition

- Approximately relate two distributions μ_1 and μ_2
- Add numeric indexes (ϵ, δ) to lifting

Want:

- Given $R \subseteq S \times T$, lift to $\langle R \rangle^{(\epsilon,\delta)} \subseteq \mathsf{Distr}(S) \times \mathsf{Distr}(T)$
- $\mu_1 \ \langle = \rangle^{(\epsilon,\delta)} \ \mu_2$ should be equivalent to $\Delta_\epsilon(\mu_1, \mu_2) \leq \delta$

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Two witnesses (Barthe and Olmedo)

There exists $\eta_L, \eta_R \in \mathsf{Distr}(S \times T)$ such that

1. $\pi_1(\eta_L) = \mu_1$ and $\pi_2(\eta_R) = \mu_2$,
2. $\eta_L(s, t), \eta_R(s, t) > 0$ only when $(s, t) \in R$,
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No witnesses (Sato)

For all subsets $A \subseteq S$, we have

$$\mu_1(A) \leq e^{\epsilon} \cdot \mu_2(R(A)) + \delta$$

Which definition is the “right” one?

Definitions support different properties and constructions

	PW-Eq	Up-to-bad	Acc. Bd.	Subset	Mapping	Adv. Comp.
1-witness	?	?	Yes	?	?	?
2-witness	Yes	Almost*	No	Almost*	Almost*	Yes
Universal	Yes	Yes	Yes	Yes	Yes	?

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Broad tradeoff: How general?

- Less general: less compositional
- More general: harder to prove properties about

Our work: $\star$-Liftings, Equivalences, and an approximate Strassen's theorem

New definition: $\star$-liftings

Generalize 2-witness lifting by adding a new point

Let $R \subseteq S \times T$ be a binary relation, and let $A^\star = A \cup \{\star\}$.
Two distributions are related by $\mu_1 \; \langle R^\star \rangle^{(\epsilon,\delta)} \; \mu_2$ if:

There exists $\eta_L, \eta_R \in \text{Distr}(S^\star \times T^\star)$ such that

1. $\pi_1(\eta_L) = \mu_1$ and $\pi_2(\eta_R) = \mu_2$,
2. $\eta_L(s,t), \eta_R(s,t) > 0$ only when $(s,t) \in R$ or $s = \star$ or $t = \star$,
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Intuition

- $\star$ is a default point for tracking “unimportant” mass

Why is $\star$-lifting a good definition?

Previously known

One-witness $\overset{(??)}{}$ Two-witness $\Rightarrow$ Universal

Why is $\star$-lifting a good definition?

Previously known

One-witness (??) Two-witness $\Longrightarrow$ Universal

$\star$-liftings unify known approximate liftings

One-witness $\Longleftrightarrow$ $\star$-lifting $\Longleftrightarrow$ Universal

Approximate version of Strassen's theorem

$\star$-liftings are equivalent to "universal" approximate liftings

Theorem

Let S, T be discrete (countable) sets, and let $R \subseteq S \times T$ be a relation. Then $\mu_1 \, \langle R^\star \rangle^{(\epsilon,\delta)} \, \mu_2$ if and only if:

for all sets $A \subseteq S$, $\mu_1(A) \leq e^\epsilon \cdot \mu_2(R(A)) + \delta$

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Proof sketch (universal lifting implies $\star$-lifting)

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Define a flow network

- Nodes

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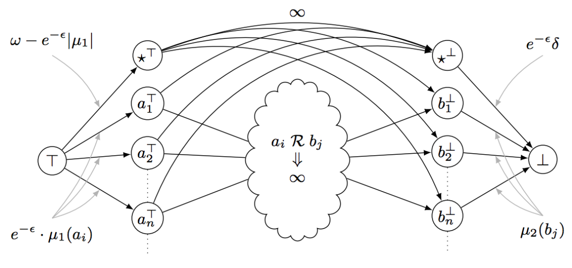

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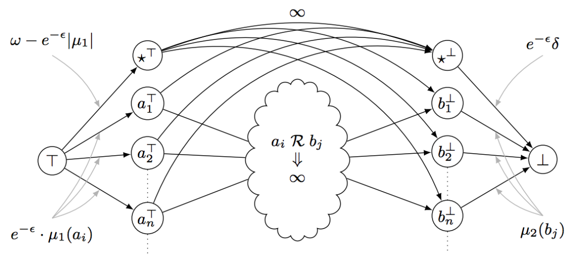

Universal lifting $\implies$ minimum cut large

- Max-flow min-cut: there is a large flow f from $\top$ to $\bot$
- Use $f(s,t)$ to recover $\star$-lifting witnesses (η_L, η_R), conclude:

$$\boxed{\mu_1 \; \langle R^\star \rangle^{(\epsilon,\delta)} \; \mu_2}$$

Other Results and Future Directions

See the paper for ...

- Further properties of $\star$-liftings
- Symmetric $\star$-liftings and advanced composition
- $\star$-liftings for f-divergences

Wrapping up: Future directions and other speculation

Open questions

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Mild speculation

$\star$-liftings are the “right” approximate version of probabilistic couplings

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