Jointly Private Convex Programming

"PrivDuDe"

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One hot summer...not enough electricity!

Solution: Turn off air-conditioning

Decide when customers get electricity

- Divide day into time slots
- Customers have values for slots
- Customers have hard minimum requirements for slots

Goal: maximize welfare

Scheduling optimization problem

Constants (Inputs to the problem)

- Customer i's value for electricity in time slot t: $v_t^{(i)} \in [0, 1]$
- Customer i's minimum requirement: $d_t^{(i)} \in [0, 1]$
- Total electricity supply in time slot t: $s_t \in \mathbb{R}$

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Variables (Outputs)

- Electricity level for user i, time t: $x_t^{(i)}$

Scheduling optimization problem

Maximize welfare

$$\max \sum_{i,t} v_t^{(i)} \cdot x_t^{(i)}$$

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...subject to constraints

- Don't exceed power supply:

$$\sum_i x_t^{(i)} \leq s_t$$

Scheduling optimization problem

Maximize welfare

$$\max \sum_{i,t} v_t^{(i)} \cdot x_t^{(i)}$$

...subject to constraints

- Don't exceed power supply:

$$\sum_{i} x_t^{(i)} \leq s_t$$

- Meet minimum energy requirements:

$$x_t^{(i)} \geq d_t^{(i)}$$

Privacy concerns

Private data

- Values $v_t^{(i)}$ for time slots
- Customer requirements $d_t^{(i)}$

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Customers shouldn't learn private data of others

More generally...

Convex program

- Want to maximize:

$$\sum_i f^{(i)}(x^{(i)}) \qquad\qquad f^{(i)} \text{ concave}$$

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- Personal constraints:

$$x^{(i)} \in S^{(i)} \qquad\qquad S^{(i)} \text{ convex}$$

More generally...

Key feature: separable

- Partition variables: Agent i's "part" of solution is $x^{(i)}$

Agent i's private data affects:

- Objective $f^{(i)}$
- Coupling constraints $g_j^{(i)}$
- Personal constraints $S^{(i)}$

Examples

- Matching LP
- d-demand fractional allocation
- Multidimensional fractional knapsack

Our results, in one slide

Theorem
Let $\varepsilon > 0$ be a privacy parameter. For a separable convex program with k coupling constraints, there is an efficient algorithm for privately finding a solution with objective at least

$$\mathrm{OPT} - O\left(\frac{k}{\varepsilon}\right),$$

and exceeding constraints by at most k/ε in total.

No polynomial dependence on number of variables

The plan today

- Convex program solution $\leftrightarrow$ equilibrium of a game
- Compute equilibrium via gradient descent
- Ensure privacy

The convex program game

The convex program two-player, zero-sum game

The players

- Primal player: plays candidate solutions $x \in S^{(1)} \times \cdots \times S^{(n)}$
- Dual player: plays dual solutions λ

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The payoff function

- Move constraints depending on multiple players (coupling constraints) into objective as penalty terms

$$\mathcal{L}(x, \lambda) = \sum_i f^{(i)}(x^{(i)}) + \sum_j \lambda_j \left(\sum_i g_j^{(i)}(x^{(i)}) - h_j \right)$$

- Primal player maximizes, dual player minimizes

Idea: Solution $\leftrightarrow$ equilibrium

Convex duality

- Optimal solution x^* gets payoff OPT versus any λ
- Optimal dual λ^* gets payoff at least $-$OPT versus any x

In game theoretic terms...

- The value of the game is OPT
- Optimal primal-dual solution (x^*, λ^*) is an equilibrium

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approximate

approximately

Finding the equilibrium

Known: techniques for finding equilibrium [FS96]

Simulated play

- First player chooses the action x_t with best payoff
- Second player uses a no-regret algorithm to select action λ_t
- Use payoff $\mathcal{L}(x_t, \lambda_t)$ to update the second player
- Repeat

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Key features

- Average of (x_t, λ_t) converges to approximate equilibrium
- Limited access to payoff data, can be made private

Gradient descent dynamics (linear case)

Idea: repeatedly go "downhill"

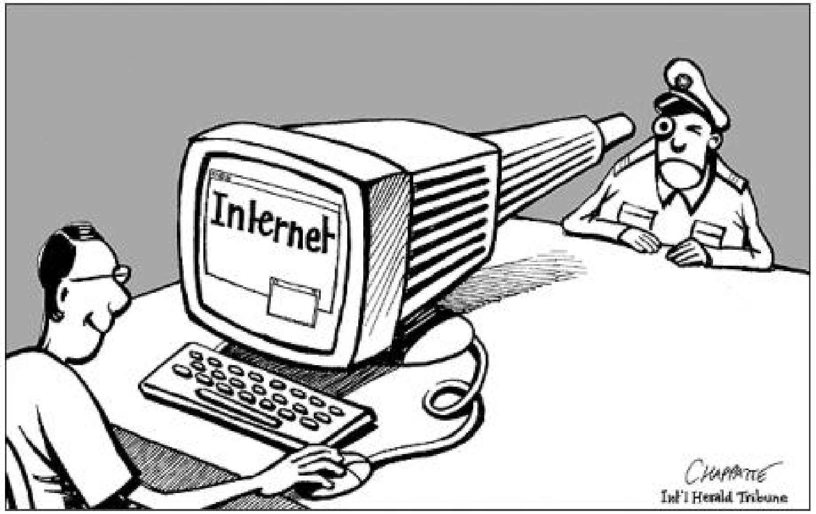
- Given primal point $x_t^{(i)}$, gradient of $\mathcal{L}(x_t, -)$ is

$$\ell_j = \sum_i g_j^{(i)} \cdot x_t^{(i)} - h_j$$

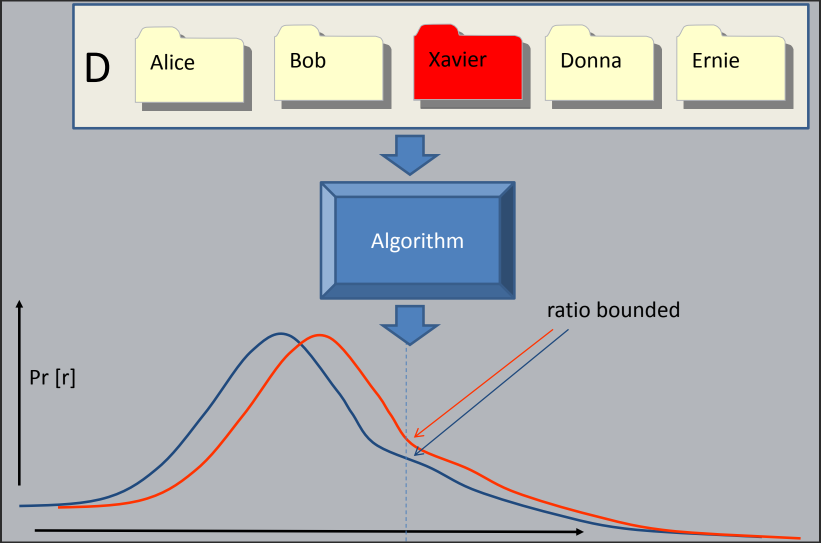
- Update:

$$\lambda_{t+1} = \lambda_t - \eta \cdot \ell$$

Achieving privacy



(Plain) Differential privacy [DMNS06]



More formally

Definition (DMNS06)

Let M be a randomized mechanism from databases to range $\mathcal{R}$, and let D, D' be databases differing in one record. M is (ε, δ)-*differentially private* if for every $S \subseteq \mathcal{R}$,

$$\Pr[M(D) \in S] \leq e^{\varepsilon} \cdot \Pr[M(D') \in S] + \delta.$$

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$$\boxed{\text{For us: too strong!}}$$

A relaxed notion of privacy [KPRU14]

Idea

- Give separate outputs to agents
- Group of agents can't violate privacy of other agents

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Definition

An algorithm $\mathcal{M} : C^n \to \Omega^n$ is (ε, δ)-*joint differentially private* if for every agent i, pair of i-neighbors $D, D' \in C^n$, and subset of outputs $S \subseteq \Omega^{n-1}$,

$$\Pr[\mathcal{M}(D)_{-i} \in S] \leq \exp(\varepsilon) \Pr[\mathcal{M}(D')_{-i} \in S] + \delta.$$

Achieving joint differential privacy

“Billboard” mechanisms

- Compute signal S satisfying standard differential privacy
- Agent i's output is a function of i's private data and S

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Lemma (Billboard lemma [HHRRW14])

Let $S : \mathcal{D} \to \mathcal{S}$ be (ε, δ)-differentially private. Let agent i have private data $D_i \in \mathcal{X}$, and let $F : \mathcal{X} \times \mathcal{S} \to \mathcal{R}$. Then the mechanism

$$M(D)_i = F(D_i, S(D))$$

is (ε, δ)-joint differentially private.

Our signal: noisy dual variables

Privacy for the dual player

- Recall gradient is

$$\ell_j = \sum_i g_j^{(i)} \cdot x_t^{(i)} - h_j$$

- May depend on private data in a low-sensitivity way

Our signal: noisy dual variables

Privacy for the dual player

- Recall gradient is

$$\ell_j = \sum_i g_j^{(i)} \cdot x_t^{(i)} - h_j$$

- May depend on private data in a low-sensitivity way
- Use Laplace mechanism to add noise, "noisy gradient":

$$\hat{\ell}_j = \sum_i g_j^{(i)} \cdot x_t^{(i)} - h_j + Lap(\Delta/\varepsilon)$$

- Noisy gradients satisfy standard differential privacy

Private action: best response to dual variables

(Joint) privacy for the primal player

- Best response problem:

$$\max_{x \in S} \mathcal{L}(x, \lambda_t) = \max_{x \in S} \sum_i f^{(i)} \cdot x^{(i)} + \sum_j \lambda_{j,t} \left(\sum_i g_j^{(i)} \cdot x^{(i)} - h_j \right)$$

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- Key point: optimization for $x^{(i)}$ depends only on λ and functions of i's private data $(S^{(i)}, f^{(i)}, g^{(i)})$

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- Output: time averages $\frac{1}{T}\sum_t x_t^{(i)}$ to agent i

Privacy guarantee

Theorem

PrivDuDe *satisfies* (ε, δ)-*joint differential privacy. The mechanism that releases just the dual variables* λ_t *satisfies* (ε, δ)-*standard differential privacy.*

Accuracy guarantee

Theorem

PrivDuDe *produces a solution x such that:*

- *it achieves objective at least $OPT - \alpha$;*
- *it satisfies all personal constraints ; and*
- *the total infeasibility over all coupling constraints is at most α ;*

where $\alpha = \tilde{O}(\sigma k \log(1/\delta)/\varepsilon)$, and σ measures the sensitivity of the convex program.

Wrapping up

See paper for...

Approximate truthfulness

Exact feasibility

Conclusion

Main ideas

- Equilibrium $\leftrightarrow$ solution to convex program
- Joint differential privacy for separable convex programs

PrivDuDe

- Approximately solve separable convex programs
- Satisfies (joint) differential privacy
- Error/infeasibility linear in number of coupling constraints

Open problems and future directions

Expanding the class of convex programs

- Can we handle something beyond separable convex programs?
- Terms depending on at most two agents?

Improving the accuracy

- Is linear dependence on number of constraints k necessary?
- What is the best dependence possible?

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