Really Naturally Linear Indexed Type Checking

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October 2, 2014

In the beginning...

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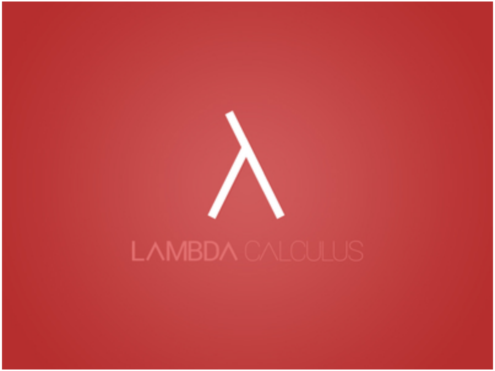

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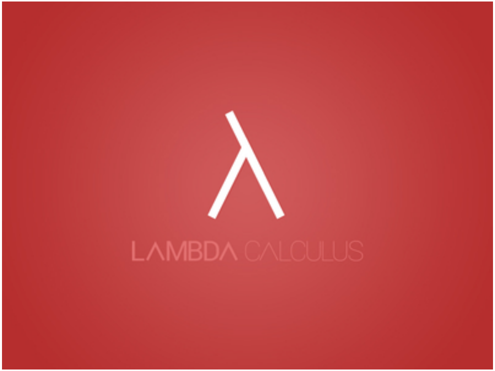

Check properties via types

- Type safety
- Parametricity
- Non-interference

More recently

Properties model quantitative information

- Numerical robustness
- Probabilistic assertions
- Differential privacy

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how robust?

how likely?

how private?

Properties not just true or false

But what about typechecking?

Typechecking quantitative languages is tricky

- May need to solve numeric constraints
- Typechecking may not be decidable
- May need heuristics to make typechecking practical

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Our goal

- Design and implement a typechecking algorithm for DFuzz, a language for verifying differential privacy

The plan today

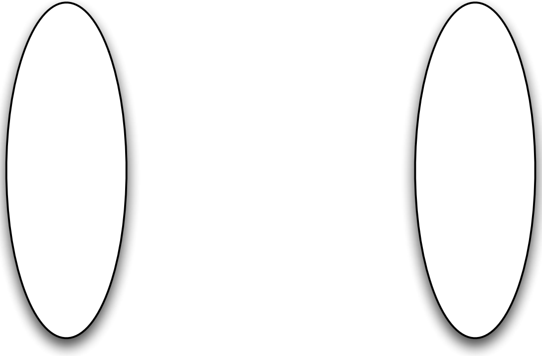
- A DFuzz crash course
- The problem with standard approaches
- Modifying the DFuzz language to ease typechecking
- Decidability and heuristics

The quantitative property

Differential privacy [DMNS06]

- Rigorous definition of privacy for randomized programs
- Idea: random noise should "conceal" an individual's data
- Quantitative: measure how private a program is
- Close connection to sensitivity analysis

R-sensitive function



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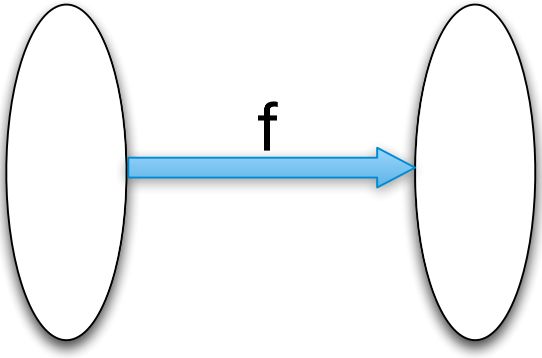

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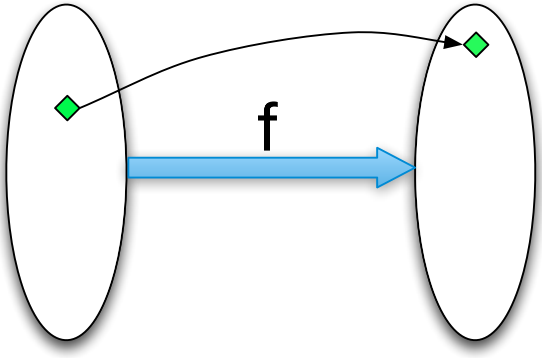

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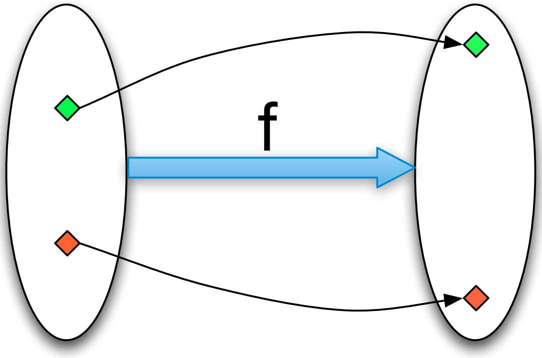

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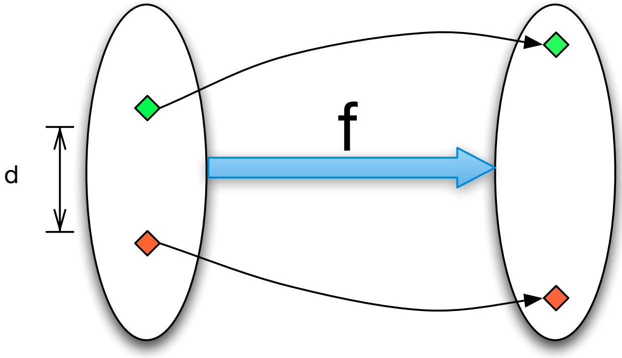

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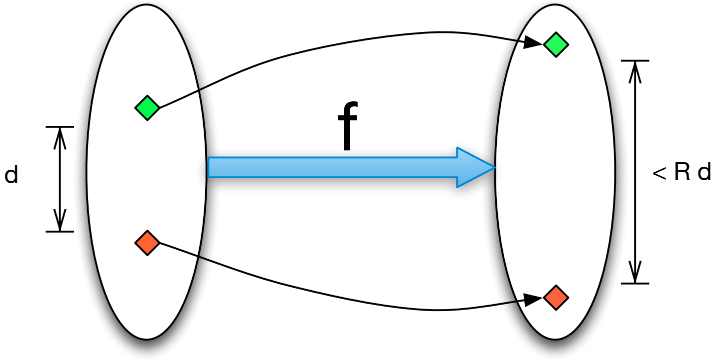

A language for differential privacy

DFuzz [GHHNP13]

- Type system for differentially private programs
- Use linear logic to model sensitivity
- Combine with (lightweight) dependent types

In a little more detail...

Types

$$\tau ::= \mathbb{N}\,[R] \;\mid \tau \oplus \tau \mid \tau \otimes \tau \mid\, !_R\, \tau \multimap \tau \mid \forall i.\, \tau$$

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Contexts

$$\Gamma ::= \cdot \mid \Gamma, x :_{[R]} \tau$$

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Typing judgment

$$\Gamma \vdash e : \tau$$

Sensitivity reading

- Functions $!_R \tau_1 \multimap \tau_2$: R-sensitive functions
- Changing input by d changes output by at most $R \cdot d$

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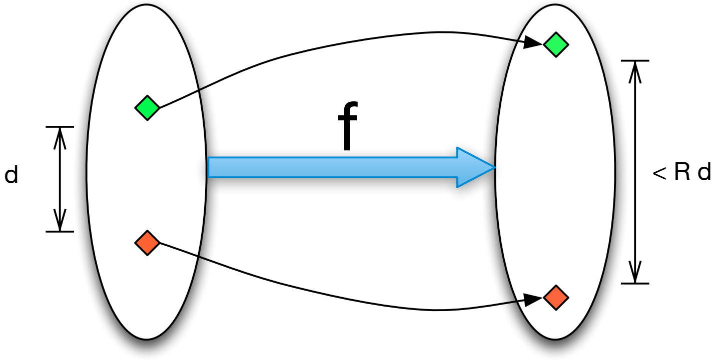

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- "A 1-sensitive function is also a 2-sensitive function"
- Subtyping: weaken sensitivity bound

$$!_R \tau \multimap \tau_2 \sqsubseteq !_{R'} \tau_1 \multimap \tau_2 \qquad \text{if} \qquad R \leq R'$$

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Grammar

$$R ::= i_{\mathbb{R}} \mid i_{\mathbb{N}} \mid \mathbb{R} \mid R + R \mid R \cdot R$$

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Sensitivity not known statically

- DFuzz is dependent!
- Sensitivity may depend on inputs (length of list, number of iterations, etc.)

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What does this mean for typechecking?

- Sensitivities are polynomials over reals and naturals
- How to check subtyping?

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compare polynomials

The typechecking problem

Assume

- Can extract type skeleton from term
- Given annotated term, compute best type

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w.r.t. subtyping

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Annotations

- We need: fully annotated argument types of all functions

$$!_{??}\ \tau_1 \multimap \tau_2$$

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no annot. annot. no annot.

- Other more minor annotations

Input

- Annotated term e
- Annotated context skeleton $\Gamma^\bullet$:

$$x :_{??} \tau$$

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no annot.

Output

- Type τ^* and context Γ with $\Gamma \vdash e : \tau^*$
- Most precise context and type (with respect to subtyping)

"Bottom-up" typechecking

- For each premise, compute best context and type
- Combine outputs from premises to get context and type

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Example: function application

$$\frac{\Gamma \vdash e_1 : !_R\sigma \multimap \tau \qquad \Delta \vdash e_2 : \sigma}{\Gamma + R \cdot \Delta \vdash e_1\ e_2 : \tau}$$

1. Given $(e_1\ e_2, \Gamma^\bullet)$

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1. Given $(e_1 \; e_2, \Gamma^\bullet)$
2. Call typechecker on $(e_1, \Gamma^\bullet)$, get $(!_R \sigma \multimap \tau, \Gamma)$

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3. Call typechecker on $(e_2, \Delta^\bullet)$, get (σ', Δ)
4. Check $\sigma' \sqsubseteq \sigma$, output $(\tau, \Gamma + R \cdot \Delta)$

A problem with the bottom-up approach

- Some DFuzz rules have form

$$\frac{\Gamma \vdash \; e_1 \; : \sigma_1 \qquad \Gamma \vdash \; e_2 \; : \sigma_2}{\Gamma \vdash \cdots : \cdots}$$

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- Running algorithm gives (σ_1, Γ_1) and (σ_2, Γ_2)
- But what context do we output?

“Minimal” context?

First try

- Have $x :_{[R_1]} \sigma$ and $x :_{[R_2]} \sigma$
- Most precise context should be $x :_{[\mathbf{max}(R_1, R_2)]} \sigma$
- But DFuzz doesn’t have $\mathbf{max}(R_1, R_2)$...

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$\boxed{\text{Max of two polynomials may not be polynomial!}}$

The idea: enrich DFuzz

EDFuzz: E(xtended) DFuzz

- Sensitivity language in DFuzz is "incomplete" for typechecking
- Add constructions like $\mathbf{max}(R_1, R_2)$ to sensitivity language
- Typecheck EDFuzz programs instead

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Relation with DFuzz

- Extension: all DFuzz programs still valid EDFuzz programs
- Preserve metatheory
- Bottom-up typechecking simple, works

How does this fix the problem?

Previously problematic rule

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- For

$$x :_{[R_1]} \sigma \in \Gamma_1 \qquad \text{and} \qquad x :_{[R_2]} \sigma \in \Gamma_2,$$

put $x :_{[\mathbf{max}(R_1, R_2)]} \sigma$ in output context

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- Return $\mathbf{max}(R_1, R_2)$ as context

Decidability?

Bad news

- Must check inequalities over reals and natural polynomials
- Subtype relation is undecidable
- Even checking validity of derivations is undecidable
- Problem for both DFuzz and EDFuzz

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Good news

- Constraint solvers are pretty good in practice
- Typical DFuzz programs rely on easy constraints

Special structure of constraints

- Allow standard (DFuzz) annotations only
- Subtyping only needs to check

  $$R \geq R^*,$$

  where R is a DFuzz sensitivity and R^* is a EDFuzz sensitivity
- R understood by standard numeric solvers
- R^* has extended terms like $\mathbf{max}(R_1, R_2), \ldots$

Checking the constraints

Idea: eliminate extended terms

- Change $R \geq \mathbf{max}(R_1^*, R_2^*)$ to

$$R \geq R_1^* \wedge R \geq R_2^*$$

- Recursively eliminate comparisons $R \geq R^*$
- Similar technique for other new sensitivity constructions

It works!

- Dispatches numeric constraints to Why3
- Typechecks examples from the DFuzz paper with no problems
- Annotation burden light on these examples

Final thoughts

Lessons learned

- Typechecking with quantitative constraints is tricky
- Numeric solvers are quite good, even for undecidable problems
- Minor details in original language can have huge effects on how easy it is to use standard solvers
- Keep typechecking in mind!

Final thoughts

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Open questions

- Does this technique of "completing" a language to ease typechecking apply to other quantitative type systems?
- Can we remove the argument type annotation in functions?

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Another example

Problematic rule

$$\frac{\Gamma \vdash e : \sigma \qquad i \text{ fresh in } \Gamma}{\Gamma \vdash \Lambda i : \kappa .\ e : \forall i : \kappa .\ \sigma}$$

Avoidance problem

- Running typechecker on $(e, \Gamma^\bullet)$ yields (σ, Γ)
- For $x :_{[R]} \sigma \in \Gamma$, want smallest R^* bigger than R but independent of i
- Again: R^* may lie outside sensitivity language
- Add construction $\mathbf{sup}(R, i)$ to EDFuzz