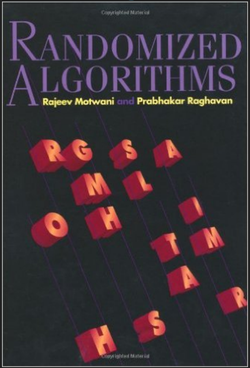
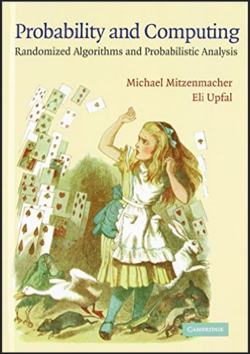
An Assertion-Based Program Logic for Probabilistic Programs

Gilles Barthe, Thomas Espitau, Marco Gaboardi,
Benjamin Grégoire, Justin Hsu, and Pierre-Yves Strub

Randomized algorithms are everywhere!





Complex programs

Algorithm 1 Joint Differentially Private Convex Solver: $\mathsf{PrivDude}(\mathcal{O}, \sigma, \tau, w, \varepsilon, \delta, \beta)$

Input: Convex problem $\mathcal{O} = (S, v, c, b)$ with n agents and k coupling constraints, gradient sensitivity bounded by σ, a dual bound τ, width bounded by w, and privacy parameters $\varepsilon > 0, \delta \in (0,1)$, confidence parameter $\beta \in (0,1)$.

Initialize:

$$\lambda_j^{(1)} := 0 \text{ for } j \in [k], \qquad T := w^2, \qquad \varepsilon' := \frac{\varepsilon\sigma}{\sqrt{8T\ln(2/\delta)}}, \qquad \delta' := \frac{\delta}{2T},$$

$$\eta := \frac{2\tau}{\sqrt{T}\left(w + \frac{1}{\varepsilon'}\log\left(\frac{Tk}{\beta}\right)\right)}, \qquad \Lambda := \{\lambda \in \mathbb{R}_+^k \mid \|\lambda\|_\infty \leq 2\tau\}.$$

for iteration $t = 1 \dots T$

for each agent $i = 0 \dots n$

Compute personal best response:

$$x_t^{(i)} := \operatorname*{argmax}_{x \in S^{(i)}} v^{(i)}(x) - \sum_{j=1}^{k} \lambda_j^{(t)} c^{(i)}(x).$$

for each constraint $j = 1 \dots k$

Compute noisy gradient:

$$\widehat{\ell}_j^{(t)} := \left(\sum_{i=0}^{n} c^{(i)}(x_t^{(i)})\right) - b_j + \mathcal{N}\left(0, \frac{2\sigma^2 \log(1.25/\delta')}{\varepsilon'^2}\right),$$

Do gradient descent update:

Complex proofs

Proof. Let ν_t denote the noise vector we have in round t, we can decompose the regret into several parts

$$
\begin{aligned}
\mathcal{R}_T &= \frac{1}{T}\sum_{t=1}^{T}\langle p_t, x_t\rangle - \frac{1}{T}\min_{p\in\mathcal{P}}\sum_{t=1}^{T}\langle p, x_t\rangle \\
&= \frac{1}{T}\sum_{t=1}^{T}\langle p_t, \widehat{x}_t\rangle - \frac{1}{T}\sum_{t=1}^{T}\langle p_t, \nu_t\rangle - \frac{1}{T}\left[\min_{p\in\mathcal{P}}\sum_{t=1}^{T}\langle p, x_t\rangle - \min_{\widehat{p}\in\mathcal{P}}\sum_{t=1}^{T}\langle \widehat{p}, \widehat{x}_t\rangle\right] - \frac{1}{T}\min_{\widehat{p}\in\mathcal{P}}\sum_{t=1}^{T}\langle \widehat{p}, \widehat{x}_t\rangle \\
&= \left[\frac{1}{T}\sum_{t=1}^{T}\langle p_t, \widehat{x}_t\rangle - \frac{1}{T}\min_{\widehat{p}\in\mathcal{P}}\sum_{t=1}^{T}\langle \widehat{p}, \widehat{x}_t\rangle\right] - \frac{1}{T}\sum_{t=1}^{T}\langle p_t, \nu_t\rangle - \frac{1}{T}\left[\min_{p\in\mathcal{P}}\sum_{t=1}^{T}\langle p, x_t\rangle - \min_{\widehat{p}\in\mathcal{P}}\sum_{t=1}^{T}\langle \widehat{p}, \widehat{x}_t\rangle\right] \\
&= \widehat{\mathcal{R}}_T - \frac{1}{T}\sum_{t=1}^{T}\langle p_t, \nu_t\rangle - \frac{1}{T}\left[\min_{p\in\mathcal{P}}\sum_{t=1}^{T}\langle p, x_t\rangle - \min_{\widehat{p}\in\mathcal{P}}\sum_{t=1}^{T}\langle \widehat{p}, \widehat{x}_t\rangle\right] \\
&\leq \widehat{\mathcal{R}}_T - \frac{1}{T}\min_{p\in\mathcal{P}}\sum_{t=1}^{T}\langle p, \nu_t\rangle - \frac{1}{T}\left[\min_{p\in\mathcal{P}}\sum_{t=1}^{T}\langle p, x_t\rangle - \min_{\widehat{p}\in\mathcal{P}}\sum_{t=1}^{T}\langle \widehat{p}, \widehat{x}_t\rangle\right].
\end{aligned}
$$

We will bound the three terms separately. By the no-regret guarantee of online gradient descent in Lemma 13, we have the following the regret guarantee w.r.t the noisy losses if we set $\eta = \frac{\|\mathcal{P}\|}{\sqrt{T}\|\widehat{\mathcal{X}}\|}$

$$
\widehat{\mathcal{R}}_T = \frac{1}{T}\sum_{t=1}^{T}\langle p_t, \widehat{x}_t\rangle - \min_{p\in\mathcal{P}}\frac{1}{T}\sum_{t=1}^{T}\langle p, \widehat{x}_t\rangle \leq \frac{\|\mathcal{P}\|^2}{2\eta T} + \frac{\eta\|\widehat{\mathcal{X}}\|^2}{2} = \frac{\|\mathcal{P}\|\|\widehat{\mathcal{X}}\|}{\sqrt{T}},
$$

where $\|\mathcal{P}\|$ and $\|\widehat{\mathcal{X}}\|$ denote the bound on the ℓ_2 norm of the vectors $\{p_t\}$ and $\{\widehat{x}_t\}$ respectively.

Recall that for any random variable Y sampled from the Gaussian distribution $\mathcal{N}(0, \sigma^2)$, we

A simple randomized algorithm and property

Noisy sum

$sum \leftarrow 0;$
for $i = 1, \dots, n$ do
  $toss \xleftarrow{\$} \mathit{flip}(p);$
  $sum \leftarrow sum + toss;$
return(sum)

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Proof of correctness, on paper?

1. sum is sum of n independent p-biased coins.
2. Apply standard concentration bound, done.

Deductive verification? Not so easy.

Expectation-based approaches

- Rules manipulate single expected value/probability
- Can't directly express properties like independence
- Kozen's PPDL (1985); Morgan, McIver, Seidel's PGCL (1996)

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Program logic (assertion-based) approaches

- Use general boolean assertions on distributions
- Complex loop rules, more limited programming languages
- Chadha et al. (2007); Rand and Zdancewic (2015)

Overall goal: Narrow this gap

Work with higher-level properties as much as possible

- Minimize reasoning about single probabilities

Avoid reasoning at level of program semantics

- Side-conditions should be easy to check

Incorporate proof methods from paper proofs

- Structure the proof, abstract away unimportant details

More concretely: Our contributions

- A new program logic for probabilistic programs
- Embeddings of several specialized proof techniques
- Implementation and formalized examples

The Ellora Framework:
A Lightning Tour

The core: A program logic for probabilistic programs

The pWhile imperative language

$$c ::= x \leftarrow e \mid x \xleftarrow{\$} d \mid c; c \mid \text{if } e \text{ then } c \text{ else } c \mid \text{while } e \text{ do } c$$

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Sample from primitive distributions

- Biased coin flips, uniform distribution, …
- Geometric distribution, Laplace distribution, …

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Commands transform (sub-)distributions over memories

- Distribution over inputs $\mapsto$ Distribution over outputs

Assertion language: two layers

State assertions: model memories

$$\phi, \psi \quad ::= \quad e = e' \mid e < e' \mid \ldots$$

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$$\Phi, \Psi \quad ::= \quad \mathbb{E}[e] = \mathbb{E}[e'] \mid \mathbb{E}[e] < \mathbb{E}[e'] \mid \dots$$

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Examples: defined notation

$$\mathbb{P}[\phi] \triangleq \mathbb{E}[1_\phi] \qquad\qquad \Box\phi \triangleq \mathbb{P}[\phi] = 1$$

Proof system

Typical program logic judgment

$$\{\Phi\}\ c\ \{\Psi\}$$

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System rules

$$\frac{\eta_0 \Rightarrow \eta_1 \qquad \{\eta_1\}\ s\ \{\eta_2\} \qquad \eta_2 \Rightarrow \eta_3}{\{\eta_0\}\ s\ \{\eta_3\}}\ [\textsc{Conseq}] \qquad \frac{\forall x : T.\ \{\eta\}\ s\ \{\eta'\}}{\{\exists x : T.\ \eta\}\ s\ \{\eta'\}}\ [\textsc{Exists}]$$

$$\frac{}{\{\eta\}\ \textbf{abort}\ \{\Box\bot\}}\ [\textsc{Abort}] \qquad \frac{\eta' \triangleq \eta[\![x \leftarrow e]\!]}{\{\eta'\}\ x \leftarrow e\ \{\eta\}}\ [\textsc{Assgn}] \qquad \frac{}{\{\eta\}\ \textbf{skip}\ \{\eta\}}\ [\textsc{Skip}]$$

$$\frac{\eta' \triangleq \eta[\![x \xleftarrow{\$} g]\!]}{\{\eta'\}\ x \xleftarrow{\$} g\ \{\eta\}}\ [\textsc{Sample}] \qquad \frac{\{\eta_0\}\ s_1\ \{\eta_1\} \qquad \{\eta_1\}\ s_2\ \{\eta_2\}}{\{\eta_0\}\ s_1; s_2\ \{\eta_2\}}\ [\textsc{Seq}]$$

$$\frac{\{\eta_1 \wedge \Box e\}\ s_1\ \{\eta_1'\} \qquad \{\eta_2 \wedge \Box\neg e\}\ s_2\ \{\eta_2'\}}{\{(\eta_1 \wedge \Box e) \oplus (\eta_2 \wedge \Box\neg e)\}\ \textbf{if}\ e\ \textbf{then}\ s_1\ \textbf{else}\ s_2\ \{\eta_1' \oplus \eta_2'\}}\ [\textsc{Cond}]$$

$$\frac{\{\eta_1\}\ s\ \{\eta_1'\} \qquad \{\eta_2\}\ s\ \{\eta_2'\}}{\{\eta_1 \oplus \eta_2\}\ s\ \{\eta_1' \oplus \eta_2'\}}\ [\textsc{Split}]$$

How to reason about loops?

Well-known pitfall: naive rule unsound!

- Always have:

$$\{\mathbb{P}[\top] = 1\} \text{ skip } \{\mathbb{P}[\top] = 1\}$$

- But not:

$$\{\mathbb{P}[\top] = 1\} \text{ while true do skip } \{\mathbb{P}[\top] = 1\}$$

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Tradeoff

Generality of invariants/allowed termination behavior

Our solution: A family of loop rules

$$\frac{\{\Phi \wedge \Box b\}\ c\ \{\Phi\}}{\{\Phi\}\ \mathsf{while}\ b\ \mathsf{do}\ c\ \{\Phi \wedge \Box \neg b\}}$$

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- Invariant Φ: arbitrary predicate

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Loop: Terminates with probability 1

- Invariant Φ: “topologically closed” (e.g., $\mathbb{P}[\phi] = 1/2$)

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- Invariant Φ: “topologically closed” (e.g., $\mathbb{P}[\phi] = 1/2$)

Loop: Arbitrary termination

- Invariant Φ: “downwards closed” (e.g., $\mathbb{P}[\phi] < 1/2$)

Adding to the Toolbox:
Specialized Proof Techniques

Two common properties in paper proofs

Probabilistic independence

- In our assertions:

$$e \# e' \triangleq \forall a, b.\, \mathbb{P}[e = a \land e' = b] = \mathbb{P}[e = a] \cdot \mathbb{P}[e' = b]$$

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Distribution laws

- In our assertions:

$$e \sim \textsf{Unif}(A) \triangleq \forall a \in A.\, \mathbb{P}[e = a] = 1/|A|$$

Reasoning about independence and distribution laws

Useful facts about independence

$$(e_1, e_2) \# e_3 \implies (e_1 \# e_3) \wedge (e_2 \# e_3)$$

Combining independence and uniformity

$$e \sim \mathsf{Unif}(A) \wedge e' \sim \mathsf{Unif}(A') \wedge (e \# e') \implies (e, e') \sim \mathsf{Unif}(A \times A')$$

Incorporating this reasoning in Ellora

Build a program logic IL around these assertions,
soundness by embedding into core program logic.

Other tools available in Ellora

Prior work: union bound logic [ICALP 2016]

- Designed for proving proeprties of the form $\mathbb{P}[\phi] < \beta$

Precondition calculus

- Similar to Morgan and McIver's weakest pre-expectations
- Defined on syntax of assertions

Implementation and Formalized Examples

Implementation

Part of the EasyCrypt system

- Tactic-based proofs, SMT support

Formalization of basic discrete probability theory

- Definitions: independence, basic distributions, ...
- Theorems: Markov inequality, Chernoff bound, ...

Examples: Nine verified algorithms

Name	Lines of Code	Lines of Proof
`hypercube`	100	1140
`coupon`	27	184
`vertex-cover`	30	61
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Hypergraph network

- Nodes: $\{0, 1\}^d$
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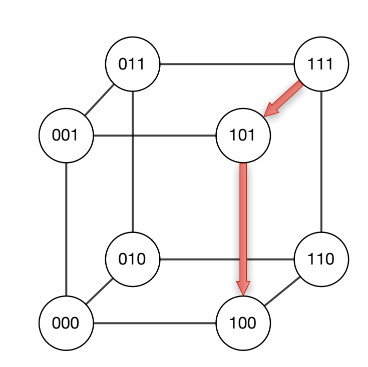

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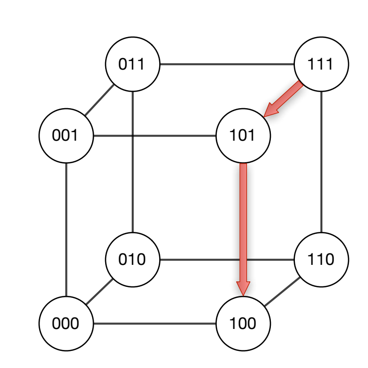

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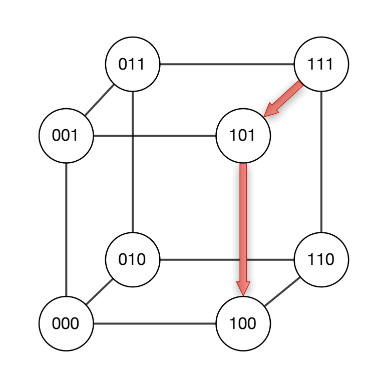

Show: with high probability, routes all 2^d packets in $O(d)$ steps

Future Directions and Open Design Questions

The story so far

Ellora

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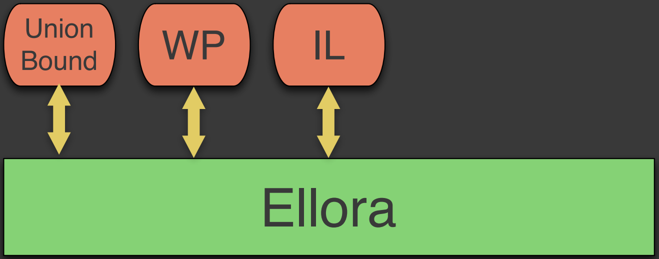

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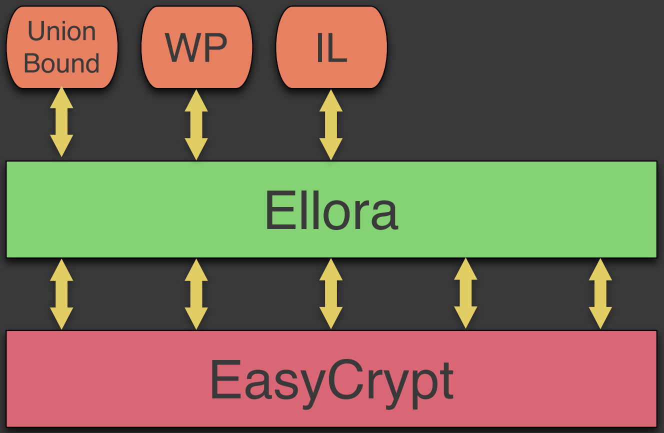

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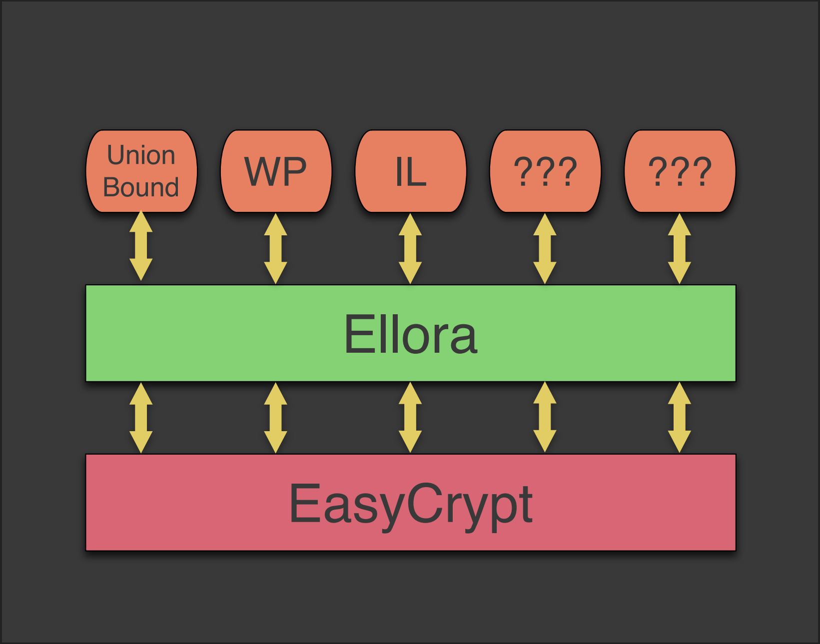

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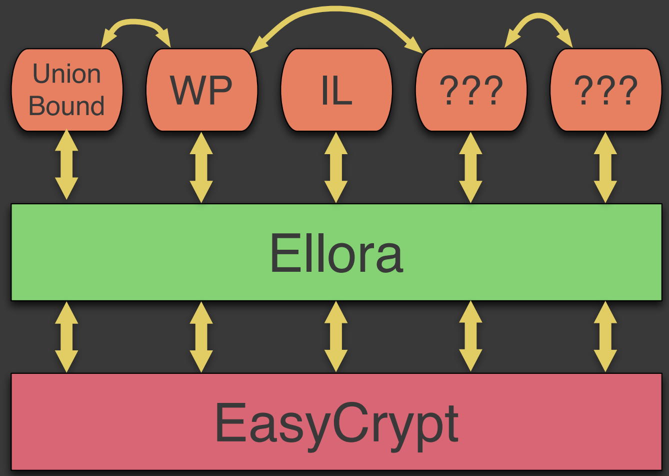

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Should reasoning be code-directed?

- Maybe easier: lift random sampling instructions out

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