Verifying Probabilistic Properties with Probabilistic Couplings

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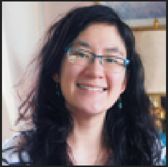
Work with brilliant collaborators





What Are Probabilistic “Relational Properties”?

Today's target properties

Probabilistic

- Programs can take random samples (flip coins)
- Map (single) input value to a distribution over outputs

Relational

- Compare two executions of a program (or: two programs)
- Describe outputs (distributions) from two related inputs
- Also known as 2-properties, or hyperproperties

Examples throughout computer science...

Security and privacy

- Indistinguishability
- Differential privacy

Machine learning

- Uniform stability

... and beyond

- Incentive properties (game theory/mechanism design)
- Convergence and mixing (probability theory)

Challenges for formal verification

Reason about two sources of randomness

- Two executions may behave very differently
- Completely different control flow (even for same program!)

Quantitative reasoning

- Target properties describe distributions
- Probabilities, expected values, etc.
- Very messy for formal reasoning

Today: Combine two ingredients

Probabilistic Couplings

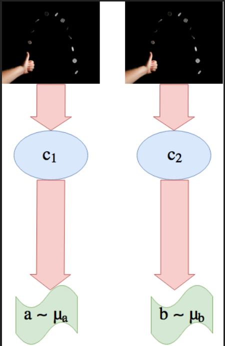
$+$

Relational Program Logics

Probabilistic Couplings and “Proof by Coupling”

Given: programs c_1 and c_2, each taking 10 coin flips

Experiment #1



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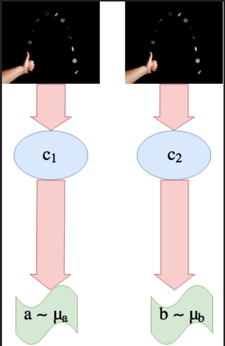
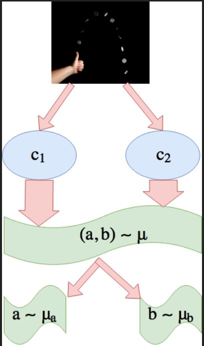


Experiment #2



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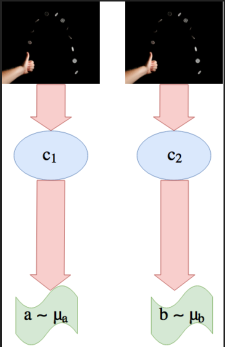

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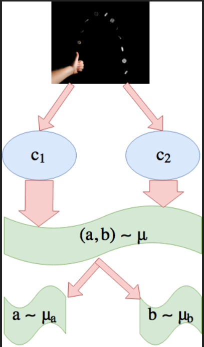

Distributions equal in Experiment #1

$\Longleftrightarrow$

Distributions equal in Experiment #2

Given: programs c_1 and c_2, each taking 10 coin flips

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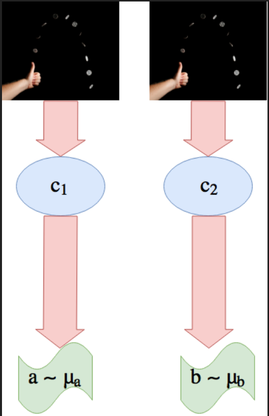

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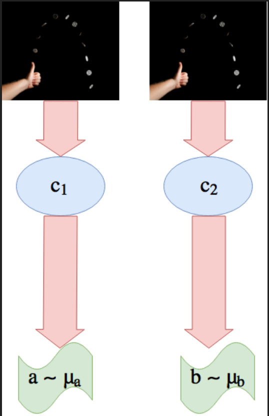
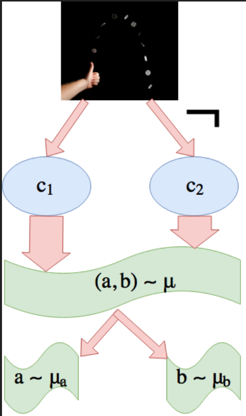


Experiment #2



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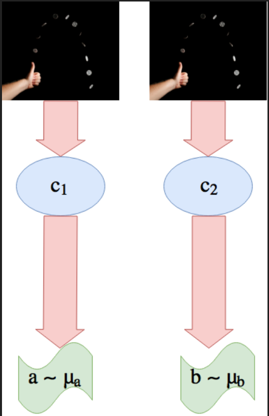

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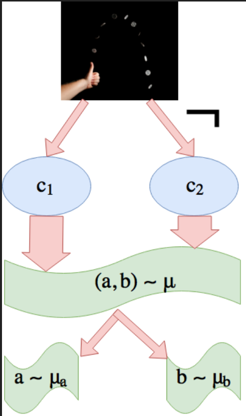


Distributions equal in Experiment #1

$\iff$

Distributions equal in Experiment #2

Why “pretend” two executions share randomness?

Easier to reason about one source of randomness

- Fewer possible executions
- Pairs of coordinated executions follow similar control flow

Reduce quantitative reasoning

- Reason on (non-probabilistic) relations between samples
- Don’t need to work with raw probabilities (messy)

A bit more precisely...

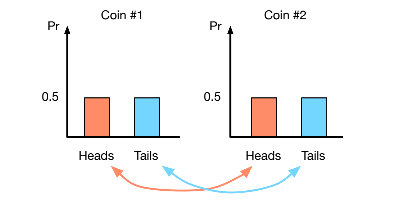
A coupling of two distributions $\mu_1, \mu_2 \in \mathsf{Distr}(A)$ is a joint distribution $\mu \in \mathsf{Distr}(A \times A)$ with $\pi_1(\mu) = \mu_1$ and $\pi_2(\mu) = \mu_2$.

A bit more precisely...

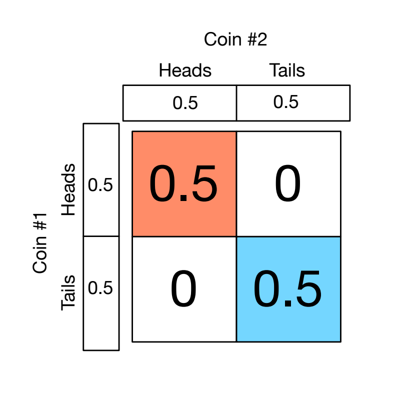
A coupling of two distributions $\mu_1, \mu_2 \in \mathsf{Distr}(A)$ is a joint distribution $\mu \in \mathsf{Distr}(A \times A)$ with $\pi_1(\mu) = \mu_1$ and $\pi_2(\mu) = \mu_2$.

A coupling models two distributions
sharing one source of randomness

For example

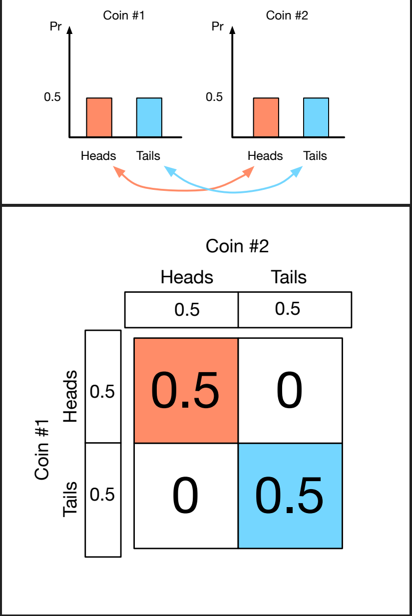


Coin #1
Pr
0.5
Heads
Tails
Coin #2
Pr
0.5
Heads
Tails



Coin #2
Heads
Tails
0.5
0.5
Coin #1
Heads
Tails
0.5
0.5
0.5
0
0
0.5

For example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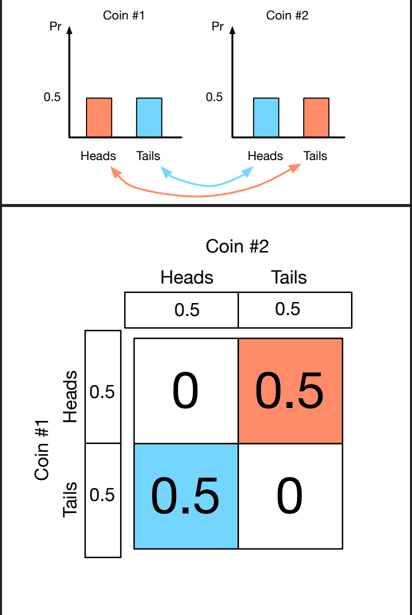
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Why are couplings interesting for verification?

Existence of a coupling* can imply a property of two distributions

If there exists a coupling of (μ_1, μ_2) where:

then:

Two coupled samples differ with small probability

μ_1 is “close” to μ_2

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Two coupled samples differ with small probability	μ_1 is “close” to μ_2
Two coupled samples are always equal	μ_1 is “equal” to μ_2

If there exists a coupling of (μ_1, μ_2) where:	then:
Two coupled samples differ with small probability	μ_1 is "close" to μ_2
Two coupled samples are always equal	μ_1 is "equal" to μ_2
First coupled sample is always larger than second sample	μ_1 "dominates" μ_2

Our plan to verify these properties

Three easy steps

1. Start from two given programs
2. Show that for two related inputs, there exists a coupling of the output distributions with certain properties
3. Conclude relational property of program(s)

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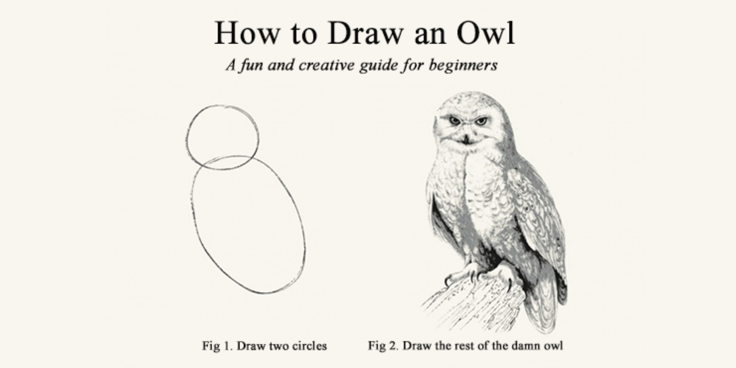

Show existence of a coupling by constructing it

A coupling proof is a recipe
for constructing a coupling

Show existence of a coupling by constructing it

A coupling proof is a recipe for constructing a coupling

1. Specify: How to couple pairs of intermediate samples
2. Deduce: Relation between final coupled samples
3. Conclude: Property about two original distributions

Probabilistic Relational Program Logics

Make statements about imperative programs

Imperative language WHILE

$$c ::= \mathsf{skip} \mid x \leftarrow e \mid \mathsf{if}\ b\ \mathsf{then}\ c\ \mathsf{else}\ c' \mid c;\ c' \mid \mathsf{while}\ b\ \mathsf{do}\ c$$

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Semantics: WHILE programs transform memories

- Variables: Fixed set $\mathcal{X}$ of program variable names
- Memories $\mathcal{M}$: functions from $\mathcal{X}$ to values $\mathcal{V}$ (e.g., 42)
- Interpret each command c as a memory transformer:

$$[\![c]\!] : \mathcal{M} \to \mathcal{M}$$

Program logics (Floyd-Hoare logics)

Logical judgments look like this

$$\{P\} \;\; c \;\; \{Q\}$$

Interpretation

- Program c, WHILE program (e.g., $x \leftarrow y; y \leftarrow y+1$)
- Precondition P, formula over $\mathcal{X}$ (e.g., $y \geq 0$)
- Postcondition Q, formula over $\mathcal{X}$ (e.g., $x \geq 0 \land y \geq 0$)

If P holds before running c, then Q holds after running c

Probabilistic Relational Hoare Logic (pRHL) [BGZ-B]

Previously

- Inspired by Benton's Relational Hoare Logic
- Foundation of the EasyCrypt system
- Verified security of many cryptographic schemes

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New interpretation

pRHL is a logic for formal proofs by coupling

Language and judgments

The pWhile imperative language

$$c ::= \mathsf{skip} \mid x \leftarrow e \mid x \xleftarrow{\$} d \mid \mathsf{if}\ e\ \mathsf{then}\ c\ \mathsf{else}\ c \mid c; c \mid \mathsf{while}\ e\ \mathsf{do}\ c$$

Language and judgments

The pWhile imperative language

$$c ::= \mathsf{skip} \mid x \leftarrow e \mid \boxed{x \xleftarrow{\$} d} \mid \mathsf{if}\ e\ \mathsf{then}\ c\ \mathsf{else}\ c \mid c; c \mid \mathsf{while}\ e\ \mathsf{do}\ c$$

Semantics of pWhile programs

- Input: a single memory (assignment to variables)
- Output: a distribution over memories
- Interpret each command c as:

$$[\![c]\!] : \mathcal{M} \to \mathsf{Distr}(\mathcal{M})$$

Basic pRHL judgments

$$\{P\} \;\; c_1 \sim c_2 \;\; \{Q\}$$

- P and Q are formulas over program variables
- Labeled program variables: x_1, x_2
- P is precondition, Q is postcondition

Interpreting the judgment

Logical judgments in pRHL look like this

$$\{P\} \;\; c_1 \sim c_2 \;\; \{Q\}$$

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Interpreting pre- and post-conditions

- As usual, P is a relation on two memories
- Q interpreted as a relation $\langle Q \rangle$ on memory distributions

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Logical judgments in pRHL look like this

$$\{P\}\ c_1 \sim c_2\ \{Q\}$$

Interpreting pre- and post-conditions

- As usual, P is a relation on two memories
- Q interpreted as a relation $\langle Q \rangle$ on memory distributions

Definition (Valid pRHL judgment)

For any pair of related inputs $(m_1, m_2) \in [\![P]\!]$, there exists a coupling $\mu \in \mathsf{Distr}(\mathcal{M} \times \mathcal{M})$ of the output distributions $([\![c_1]\!]m_1, [\![c_2]\!]m_2)$ such that $\mathsf{supp}(\mu) \subseteq [\![Q]\!]$.

Encoding couplings with PRHL theorems

$$\{P\} \ c_1 \sim c_2 \ \{o_1 = o_2\}$$

Interpretation

If two inputs satisfy P, there exists a coupling of the output distributions where the coupled samples have equal o

Encoding couplings with PRHL theorems

$$\{P\} \;\; c_1 \sim c_2 \;\; \{o_1 = o_2\}$$

Interpretation

If two inputs satisfy P, there exists a coupling of the output distributions where the coupled samples have equal o

This implies:

If two inputs satisfy P, the distributions of o are equal

Encoding couplings with pRHL theorems

$$\{P\}\ c_1 \sim c_2\ \{o_1 \geq o_2\}$$

This implies:

If two inputs satisfy P, then the first distribution of o stochastically dominates the second distribution of o

Proving Judgments:

The Proof System of pRHL

More convenient way to prove judgments

Inference rules describe:

- Judgments that are always true (axioms)
- How to prove judgment for a program by combining judgments for components

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Example: sequential composition rule

Given: $\{P\}\ c_1\ \{Q\}$ and $\{Q\}\ c_2\ \{R\}$

More convenient way to prove judgments

Inference rules describe:

- Judgments that are always true (axioms)
- How to prove judgment for a program by combining judgments for components

Example: sequential composition rule

$$\frac{\text{Given:} \quad \{P\}\ c_1\ \{Q\} \quad \text{and} \quad \{Q\}\ c_2\ \{R\}}{\text{Conclude:} \quad \{P\}\ c_1\ ;\ c_2\ \{R\}}$$

Reading the rules: introduce couplings

$$\frac{}{\vdash \{\ \}\ \ x_1 \xleftarrow{\$} \mathit{flip} \sim x_2 \xleftarrow{\$} \mathit{flip}\ \ \{x_1 = x_2\}}$$

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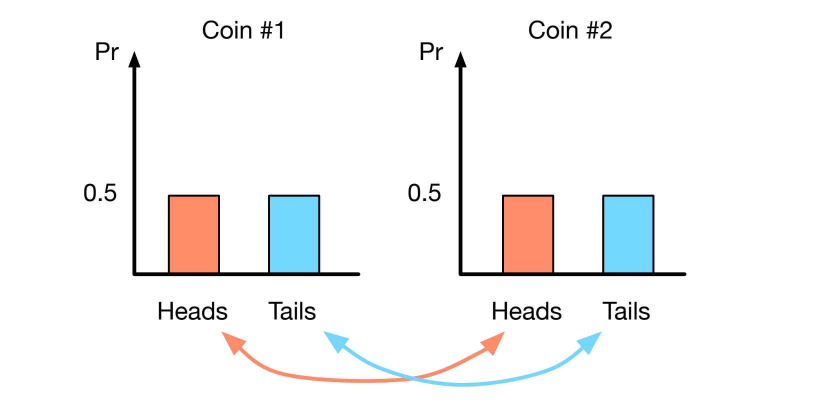

Reading the rules: introduce couplings

$$\frac{}{\vdash \{\ \}\ \ x_1 \xleftarrow{\$} \mathit{flip} \sim x_2 \xleftarrow{\$} \mathit{flip}\ \ \{x_1 \neq x_2\}}$$

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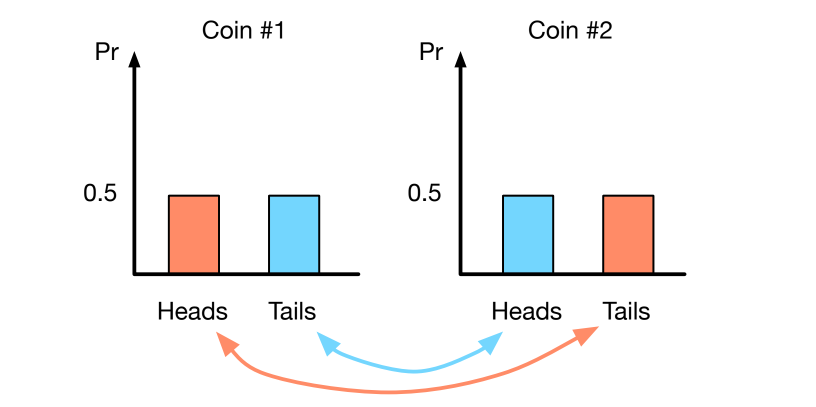

Reading the rules: combine couplings

$$\frac{\vdash \{P\}\ c_1 \sim c_2\ \{Q\} \qquad \vdash \{Q\}\ c_1' \sim c_2'\ \{R\}}{\vdash \{P\}\ c_1; c_1' \sim c_2; c_2'\ \{R\}}$$

Reading the rules: combine couplings

$$\frac{\vdash \{P\}\ c_1 \sim c_2\ \{Q\} \qquad \vdash \{Q\}\ c_1' \sim c_2'\ \{R\}}{\vdash \{P\}\ c_1; c_1' \sim c_2; c_2'\ \{R\}}$$

$$\boxed{\text{Sequence couplings}}$$

Reading the rules: combine couplings

$$\frac{\vdash \{P \wedge S\}\ c_1 \sim c_2\ \{Q\} \qquad \vdash \{P \wedge \neg S\}\ c_1 \sim c_2\ \{Q\}}{\vdash \{P\}\ c_1 \sim c_2\ \{Q\}}$$

Reading the rules: combine couplings

$$\frac{\vdash \{P \wedge S\}\ c_1 \sim c_2\ \{Q\} \qquad \vdash \{P \wedge \neg S\}\ c_1 \sim c_2\ \{Q\}}{\vdash \{P\}\ c_1 \sim c_2\ \{Q\}}$$

$\boxed{\text{Select couplings}}$

Reading the rules: combine couplings

$$\frac{\vdash \{P \wedge e_1 \wedge e_2\}\ \ c_1 \sim c_2\ \ \{P\} \qquad \models P \to e_1 = e_2}{\vdash \{P\}\ \ \textsf{while}\ e_1\ \textsf{do}\ c_1 \sim \textsf{while}\ e_2\ \textsf{do}\ c_2\ \ \{P \wedge (\neg e_1 \wedge \neg e_2)\}}$$

Reading the rules: combine couplings

$$\frac{\vdash \{P \land e_1 \land e_2\}\ c_1 \sim c_2\ \{P\} \qquad \models P \to e_1 = e_2}{\vdash \{P\}\ \text{while } e_1 \text{ do } c_1 \sim \text{while } e_2 \text{ do } c_2\ \{P \land (\neg e_1 \land \neg e_2)\}}$$

Repeat couplings

Reading the rules: combine couplings

$$\frac{\vdash \{P \wedge e_1 \wedge e_2\}\ c_1 \sim c_2\ \{P\} \qquad \models P \to e_1 = e_2}{\vdash \{P\}\ \text{while } e_1 \text{ do } c_1 \sim \text{while } e_2 \text{ do } c_2\ \{P \wedge (\neg e_1 \wedge \neg e_2)\}}$$

Repeat couplings

Not a rule: conjunction

$$\frac{\vdash \{P\}\ c_1 \sim c_2\ \{Q\} \qquad \vdash \{P\}\ c_1 \sim c_2\ \{R\}}{\vdash \{P\}\ c_1 \sim c_2\ \{Q \wedge R\}}$$

Not a rule: conjunction

$$\frac{\vdash \{P\}\ c_1 \sim c_2\ \{Q\} \qquad \vdash \{P\}\ c_1 \sim c_2\ \{R\}}{\vdash \{P\}\ c_1 \sim c_2\ \{Q \wedge R\}}$$

Can't compose this way

Is this just bisimulation?

More general property

- Relation need not be equivalence (bisimulation)
- Relation need not be preorder (simulation)

More general model of computation

- Probabilistic imperative programs
- State space can be infinite/parametrized

More flexible construction

- No fixed notion of a transition
- Coupling can be constructed “asynchronously”

Formal Proofs by Coupling

Ex. 1: Equivalence

Target property: equivalence

P's output distribution is the same for any two inputs

- Shows: output distribution is the same for any input
- Security: input is secret, output is encrypted

Warmup example: secrecy of one-time-pad (OTP)

The program

- Program input: a secret boolean sec
- Program output: an encrypted version of the secret

$$
\begin{array}{ll}
key \xleftarrow{\$} \mathit{flip}; & \text{// draw random key} \\
enc \leftarrow sec \oplus key; & \text{// exclusive or} \\
\text{return}(enc) & \text{// return encrypted}
\end{array}
$$

Proof by coupling

- Either sec_1, sec_2 are equal, or unequal
  1. If equal: couple sampling for key to be equal in both runs
  2. If unequal: couple sampling for key to be unequal in both runs
- Coupling ensures $enc_1 = enc_2$, hence distributions equal

Formalizing the proof in pRHL

Case 1: $sec_1 = sec_2$

- By applying identity coupling rule (general version):

$$\begin{array}{l} \{sec_1 = sec_2\} \\ key \xleftarrow{\$} flip; \\ \{key_1 = key_2\} \\ enc \leftarrow sec \oplus key \\ \{enc_1 = enc_2\} \end{array}$$

- Hence:

$$\{sec_1 = sec_2\}\ \ otp \sim otp\ \ \{enc_1 = enc_2\}$$

Formalizing the proof in pRHL

Case 2: $sec_1 \neq sec_2$

- By applying negation coupling rule (general version):

$$\begin{array}{l} \{sec_1 \neq sec_2\} \\ key \xleftarrow{\$} flip; \\ \{key_1 \neq key_2\} \\ enc \leftarrow sec \oplus key \\ \{enc_1 = enc_2\} \end{array}$$

- Hence:

$$\{sec_1 \neq sec_2\}\ \ otp \sim otp\ \ \{enc_1 = enc_2\}$$

Formalizing the proof in pRHL

Combining the cases:

$$\frac{\{sec_1 = sec_2\}\ otp \sim otp\ \{enc_1 = enc_2\} \quad \{sec_1 \neq sec_2\}\ otp \sim otp\ \{enc_1 = enc_2\}}{\{\top\}\ otp \sim otp\ \{enc_1 = enc_2\}}$$

and we are done!

Formal Proofs by Coupling

Ex. 2: Stochastic Domination

Target property: stochastic domination

Order relation on distributions

- Given: ordered set $(A, \leq_A)$
- Lift to ordering on distributions $(\text{Distr}(A), \leq_{sd})$

For naturals $(\mathbb{N}, \leq)$ ...

Two distributions $\mu_1, \mu_2 \in \text{Distr}(\mathbb{N})$ satisfy $\mu_1 \leq_{sd} \mu_2$ if

$$\text{for all } k \in \mathbb{N},\ \mu_1(\{n \mid k \leq n\}) \leq \mu_2(\{n \mid k \leq n\})$$

Proof by coupling

$ct \leftarrow 0;$
for i$= 1, \ldots, T_1$ do
  $r \xleftarrow{\$} \mathit{flip};$
  if $r =$ heads then
    $ct \leftarrow ct + 1;$
return(ct)

$ct \leftarrow 0;$
for i$= 1, \ldots, T_2$ do
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Suppose $T_1 \geq T_2$: first loop runs more

- Want to prove $\mu_1 \geq_{sd} \mu_2$

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Suppose $T_1 \geq T_2$: first loop runs more

- Want to prove $\mu_1 \geq_{sd} \mu_2$

Suffices to construct a coupling where $ct_1 \geq ct_2$

- Couple the first T_2 samples to be equal across both runs; establishes $ct_1 = ct_2$
- Take the remaining $T_1 - T_2$ samples (in the first run) to be arbitrary; preserves $ct_1 \geq ct_2$

Formalizing the proof in pRHL

$$
\begin{array}{l}
ct \leftarrow 0; \\
\text{for i} = 1, \ldots, T_1 \text{ do} \\
\quad r \xleftarrow{\$} \mathit{flip}; \\
\quad \text{if } r = \text{heads then} \\
\quad\quad ct \leftarrow ct + 1; \\
\text{return}(ct)
\end{array}
\qquad\Bigg|\qquad
\begin{array}{l}
ct \leftarrow 0; \\
\text{for i} = 1, \ldots, T_2 \text{ do} \\
\quad r \xleftarrow{\$} \mathit{flip}; \\
\quad \text{if } r = \text{heads then} \\
\quad\quad ct \leftarrow ct + 1; \\
\text{return}(ct)
\end{array}
$$

Goal: prove

$$\boxed{\vdash \{T_1 \geq T_2\} \ c_1 \sim c_2 \ \{ct_1 \geq ct_2\}}$$

Proof sketch

Step 1: Rewrite

$ct \leftarrow 0$;
for i$= 1, \dots, T_2$ do
  $r \xleftarrow{\$} \mathit{flip}$;
  if $r =$ heads then
    $ct \leftarrow ct + 1$;
for i$= T_2 + 1, \dots, T_1$ do
  $r \xleftarrow{\$} \mathit{flip}$;
  if $r =$ heads then
    $ct \leftarrow ct + 1$;
return(ct)
	$ct \leftarrow 0$;
	for i$= 1, \dots, T_2$ do
	  $r \xleftarrow{\$} \mathit{flip}$;
	  if $r =$ heads then
	    $ct \leftarrow ct + 1$;
	return(ct)

Proof sketch

$ct \leftarrow 0$;
for i$= 1, \ldots, T_2$ do
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  if $r =$ heads then
    $ct \leftarrow ct + 1$

$ct \leftarrow 0$;
for i$= 1, \ldots, T_2$ do
  $r \xleftarrow{\$} \mathit{flip}$;
  if $r =$ heads then
    $ct \leftarrow ct + 1$

Proof sketch

$ct \leftarrow 0;$
for i$= 1, \ldots, T_2$ do
  $r \xleftarrow{\$} \mathit{flip};$
  if $r =$ heads then
    $ct \leftarrow ct + 1$
	$ct \leftarrow 0;$
	for i$= 1, \ldots, T_2$ do
	  $r \xleftarrow{\$} \mathit{flip};$
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Step 2: First loop

- Use sampling rule with identity coupling: $r_1 = r_2$

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Proof sketch

$ct \leftarrow 0;$
for i$= 1, \dots, T_2$ do
  $r \xleftarrow{\$} \mathit{flip};$
  if $r =$ heads then
    $ct \leftarrow ct + 1$
	$ct \leftarrow 0;$
	for i$= 1, \dots, T_2$ do
	  $r \xleftarrow{\$} \mathit{flip};$
	  if $r =$ heads then
	    $ct \leftarrow ct + 1$

Step 2: First loop

- Use sampling rule with identity coupling: $r_1 = r_2$
- Establish loop invariant $ct_1 = ct_2$

Proof sketch

```
for i= $T_2 + 1, \ldots, T_1$ do
    $r \xleftarrow{\$} flip$;
    if $r$ = heads then
        $ct \leftarrow ct + 1$;
return($ct$)
```

return(ct)

Step 3: Second loop

- Use “one-sided” sampling rule
- Apply “one-sided” loop rule to show invariant $ct_1 \geq ct_2$

Formal Proofs by Coupling

Ex. 3: Uniformity

Simulating a fair coin flip from a biased coin

Problem setting

- Given: ability to draw biased coin flips $\mathit{flip}(p)$, $p \neq 1/2$
- Goal: simulate a fair coin flip $\mathit{flip}(1/2)$

Simulating a fair coin flip from a biased coin

Problem setting

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- Goal: simulate a fair coin flip $\mathit{flip}(1/2)$

Algorithm ("von Neumann's trick")

$x \leftarrow \mathit{true}; y \leftarrow \mathit{true};$	// initialize $x = y$
while $x = y$ do	// if equal, repeat
$\quad x \xleftarrow{\$} \mathit{flip}(p);$	// flip biased coin
$\quad y \xleftarrow{\$} \mathit{flip}(p);$	// flip biased coin
return(x)	// if not equal, return x

Simulating a fair coin flip from a biased coin

Problem setting

- Given: ability to draw biased coin flips $\mathit{flip}(p)$, $p \neq 1/2$
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$\quad y \xleftarrow{\$} \mathit{flip}(p);$	// flip biased coin
return(x)	// if not equal, return x

How to prove that the result x is unbiased (uniform)?

From existence of coupling, to uniformity

Suppose that we know there exist two couplings:

1. Under first coupling, $x_1 = \mathit{true}$ implies $x_2 = \mathit{false}$
2. Under second coupling, $x_1 = \mathit{false}$ implies $x_2 = \mathit{true}$

From existence of coupling, to uniformity

Suppose that we know there exist two couplings:

1. Under first coupling, $x_1 = \mathit{true}$ implies $x_2 = \mathit{false}$
2. Under second coupling, $x_1 = \mathit{false}$ implies $x_2 = \mathit{true}$

As a consequence:

- By (1), $\Pr[x_1 = \mathit{true}] \leq \Pr[x_2 = \mathit{false}]$
- By (2), $\Pr[x_1 = \mathit{false}] \leq \Pr[x_2 = \mathit{true}]$

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- By (2), $\Pr[x_1 = \mathit{false}] \leq \Pr[x_2 = \mathit{true}]$

But x_1 and x_2 have same distribution

- By (1), $\Pr[x_1 = \mathit{true}] \leq \Pr[x_1 = \mathit{false}]$
- By (2), $\Pr[x_1 = \mathit{false}] \leq \Pr[x_1 = \mathit{true}]$
- Hence uniform: $\Pr[x_1 = \mathit{true}] = \Pr[x_1 = \mathit{false}]$

Proof by coupling

Algorithm (“von Neumann’s trick”)

```
x ← true; y ← true;          // initialize x = y
while x = y do               // if equal, repeat
    x ←$ flip(p);            // flip biased coin
    y ←$ flip(p);            // flip biased coin
return(x)                    // if not equal, return x
```

Construct couplings such that:

1. Under first coupling, $x_1 = \mathit{true}$ implies $x_2 = \mathit{false}$
2. Under second coupling, $x_1 = \mathit{false}$ implies $x_2 = \mathit{true}$

Consider the following coupling:

- Couple sampling of x_1 to be equal to sampling of y_2
- Couple sampling of x_2 to be equal to sampling of y_1
- Resulting coupling satisfies both (1) and (2)!

Formalizing the proof in pRHL

Relate two (equivalent) versions of the program:

$x \leftarrow true; y \leftarrow true;$	$x \leftarrow true; y \leftarrow true;$
while $x = y$ do	while $x = y$ do
$\quad x \xleftarrow{\$} \mathit{flip}(p);$	$\quad y \xleftarrow{\$} \mathit{flip}(p);$
$\quad y \xleftarrow{\$} \mathit{flip}(p);$	$\quad x \xleftarrow{\$} \mathit{flip}(p);$
return(x)	return(x)

Formalizing the proof in pRHL

Relate two (equivalent) versions of the program:

$x \leftarrow true; y \leftarrow true;$ while $x = y$ do $\quad x \xleftarrow{\$} flip(p);$ $\quad y \xleftarrow{\$} flip(p);$ return(x)	$x \leftarrow true; y \leftarrow true;$ while $x = y$ do $\quad y \xleftarrow{\$} flip(p);$ $\quad x \xleftarrow{\$} flip(p);$ return(x)

Build coupling for loop bodies, then loops

- Use sampling rule with identity coupling: $x_1 = y_2$

Formalizing the proof in pRHL

Relate two (equivalent) versions of the program:

$x \leftarrow true; y \leftarrow true;$	$x \leftarrow true; y \leftarrow true;$
while $x = y$ do	while $x = y$ do
$\quad x \xleftarrow{\$} flip(p);$	$\quad y \xleftarrow{\$} flip(p);$
$\quad y \xleftarrow{\$} flip(p);$	$\quad x \xleftarrow{\$} flip(p);$
return(x)	return(x)

Build coupling for loop bodies, then loops

- Use sampling rule with identity coupling: $x_1 = y_2$

Formalizing the proof in pRHL

Relate two (equivalent) versions of the program:

$$
\begin{array}{l|l}
\begin{array}{l}
x \leftarrow \mathit{true}; y \leftarrow \mathit{true}; \\
\text{while } x = y \text{ do} \\
\quad x \xleftarrow{\$} \mathit{flip}(p); \\
\quad y \xleftarrow{\$} \mathit{flip}(p); \\
\text{return}(x)
\end{array}
&
\begin{array}{l}
x \leftarrow \mathit{true}; y \leftarrow \mathit{true}; \\
\text{while } x = y \text{ do} \\
\quad y \xleftarrow{\$} \mathit{flip}(p); \\
\quad x \xleftarrow{\$} \mathit{flip}(p); \\
\text{return}(x)
\end{array}
\end{array}
$$

Build coupling for loop bodies, then loops

- Use sampling rule with identity coupling: $x_1 = y_2$
- Use sampling rule with identity coupling: $y_1 = x_2$

Formalizing the proof in pRHL

Relate two (equivalent) versions of the program:

$x \leftarrow \mathit{true}; y \leftarrow \mathit{true};$	$x \leftarrow \mathit{true}; y \leftarrow \mathit{true};$
while $x = y$ do	while $x = y$ do
$\quad x \xleftarrow{\$} \mathit{flip}(p);$	$\quad y \xleftarrow{\$} \mathit{flip}(p);$
$\quad y \xleftarrow{\$} \mathit{flip}(p);$	$\quad x \xleftarrow{\$} \mathit{flip}(p);$
return(x)	return(x)

Build coupling for loop bodies, then loops

- Use sampling rule with identity coupling: $x_1 = y_2$
- Use sampling rule with identity coupling: $y_1 = x_2$

Formalizing the proof in pRHL

Relate two (equivalent) versions of the program:

$x \leftarrow true; y \leftarrow true;$ while $x = y$ do $\quad x \xleftarrow{\$} flip(p);$ $\quad y \xleftarrow{\$} flip(p);$ return(x)	$x \leftarrow true; y \leftarrow true;$ while $x = y$ do $\quad y \xleftarrow{\$} flip(p);$ $\quad x \xleftarrow{\$} flip(p);$ return(x)

Build coupling for loop bodies, then loops

- Use sampling rule with identity coupling: $x_1 = y_2$
- Use sampling rule with identity coupling: $y_1 = x_2$
- Use loop rule with invariant:

$$(x_1 = y_1 \rightarrow x_1 = y_2) \wedge (x_1 \neq y_1 \rightarrow x_1 \neq x_2)$$

Formalizing the proof in pRHL

Relate two (equivalent) versions of the program:

$x \leftarrow true; y \leftarrow true;$
while $x = y$ do
    $x \xleftarrow{\$} flip(p);$
    $y \xleftarrow{\$} flip(p);$
return(x)
	$x \leftarrow true; y \leftarrow true;$
	while $x = y$ do
	    $y \xleftarrow{\$} flip(p);$
	    $x \xleftarrow{\$} flip(p);$
	return(x)

Build coupling for loop bodies, then loops

- Use sampling rule with identity coupling: $x_1 = y_2$
- Use sampling rule with identity coupling: $y_1 = x_2$
- Use loop rule with invariant:

$$(x_1 = y_1 \to x_1 = y_2) \wedge (x_1 \neq y_1 \to x_1 \neq x_2)$$

Wrapping Up

Variations and extensions

Approximate couplings

- Prove differential privacy as approximate equivalence
- Coming up next in Marco's tutorial!

Expectation couplings

- Prove quantitative bounds on distance b/t distributions
- MC convergence, stability of ML, path coupling, ...
- Program logic: https://arxiv.org/abs/1708.02537
- Pre-expectation calculus: https://arxiv.org/abs/1901.06540

Automation

- Encode search for coupling proofs as a synthesis problem
- Coupling proofs: https://arxiv.org/abs/1804.04052
- Approximate couplings: https://arxiv.org/abs/1709.05361

References

Relational reasoning via probabilistic coupling

- Initial connection between couplings and pRHL (LPAR 2015)
- arXiv: https://arxiv.org/abs/1509.03476

Coupling proofs are probabilistic product programs

- Extract product programs from pRHL proofs (POPL 2016)
- arXiv: https://arxiv.org/abs/1607.03455

Proving uniformity and independence by self-composition and coupling

- Coupling proofs for non-relational properties (LPAR 2017)
- arXiv: https://arxiv.org/abs/1701.06477

Verifying Probabilistic Properties with Probabilistic Couplings

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