Coupling Proofs Are Probabilistic Product Programs

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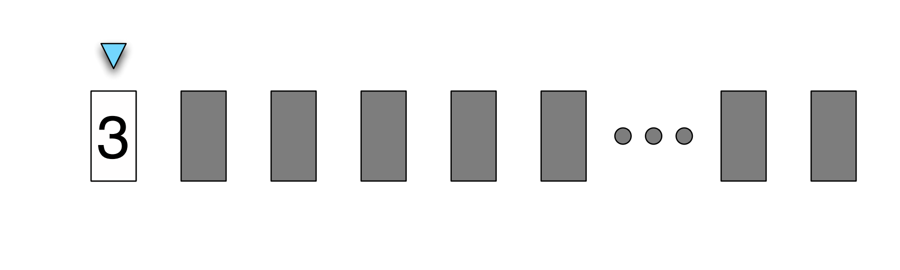
A simple card-flipping process

Setup

- Input: position in $\{1, \ldots, 9\}$
- Repeat:
  - Draw uniformly random card $\in \{1, \ldots, 9\}$
  - Go forward that many steps
- Output last position before crossing 100

In pictures



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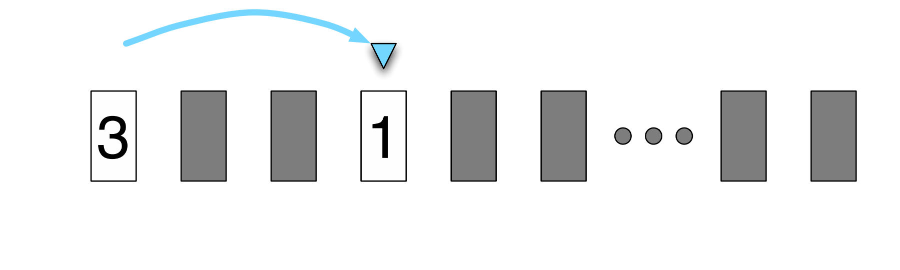

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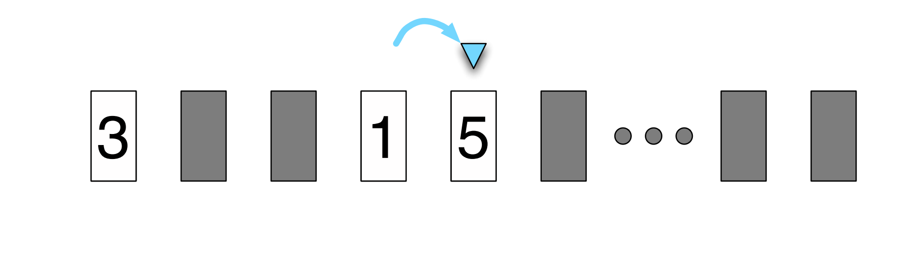

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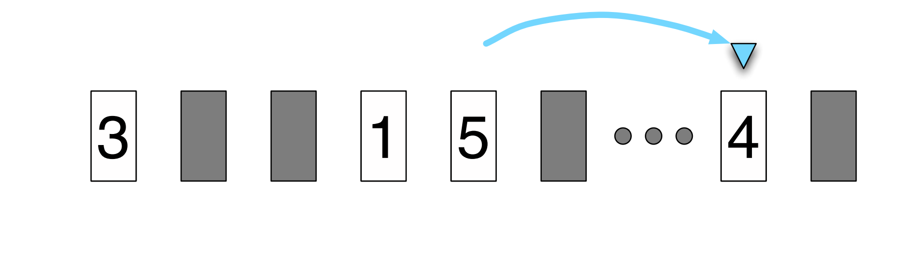

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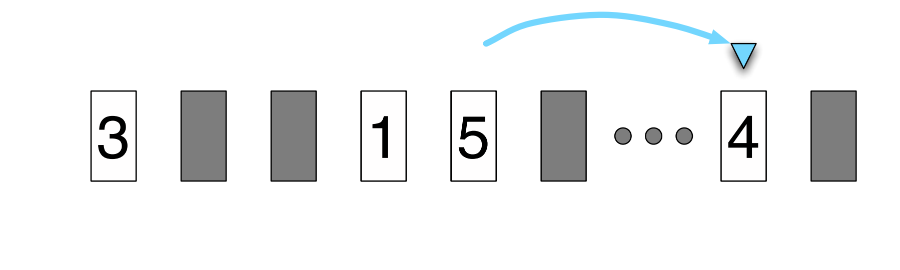
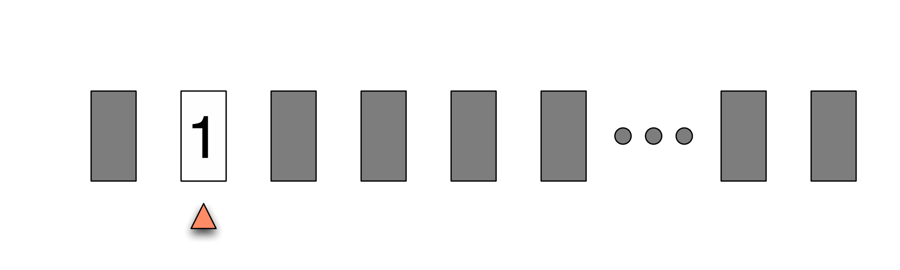


Output last position: 99

Starting at a different position



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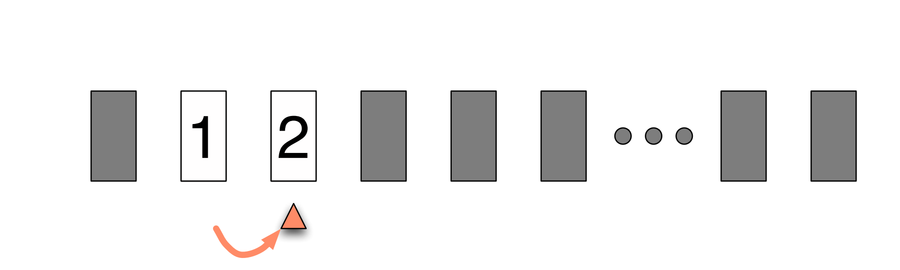

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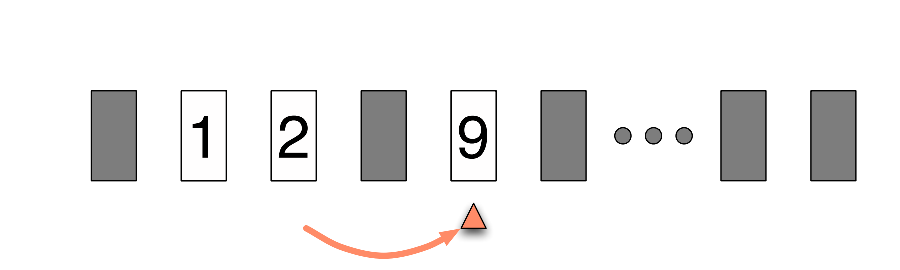

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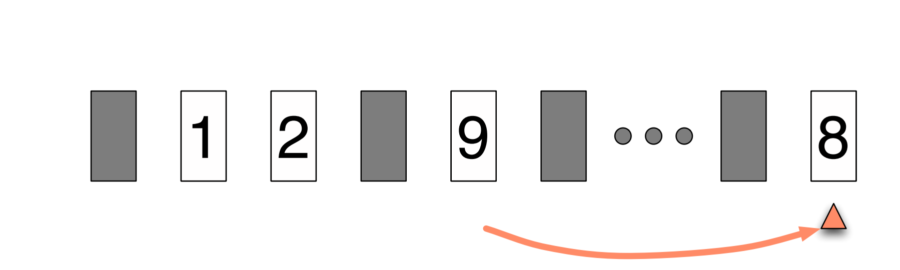

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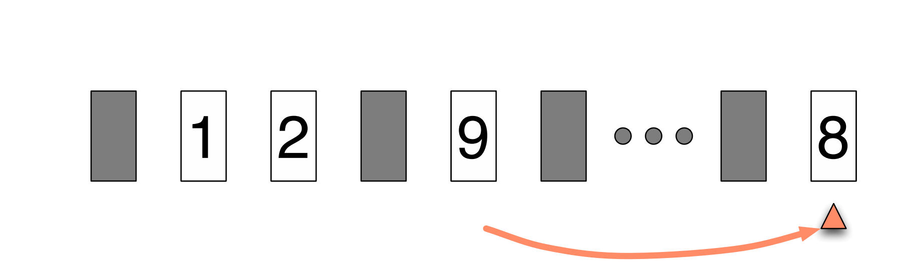
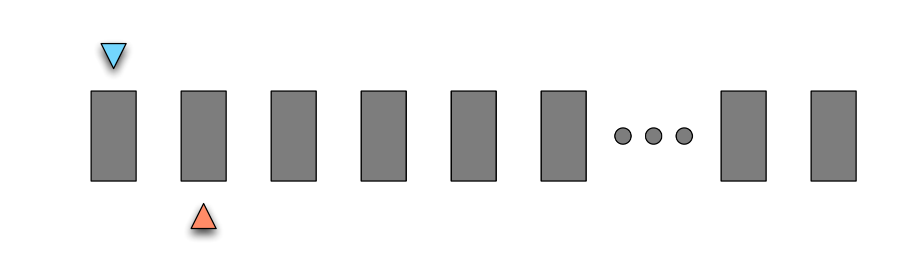


How close are the two output distributions?

Combine first process and second process



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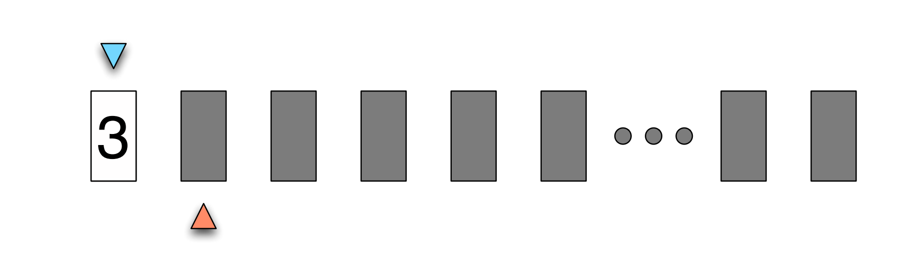

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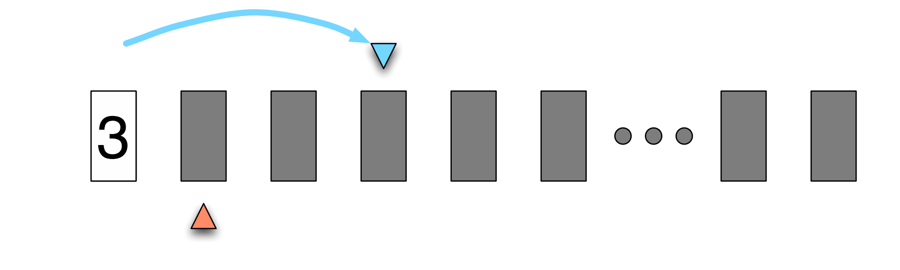

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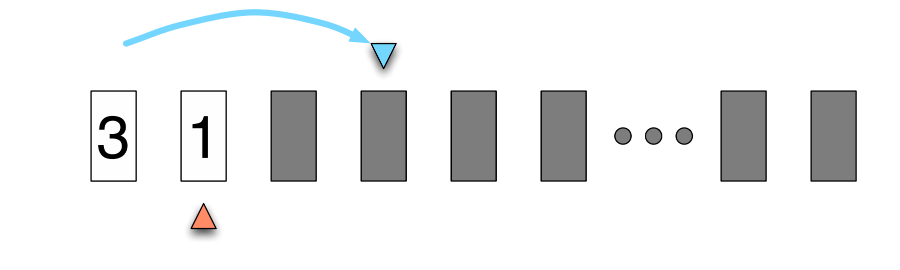

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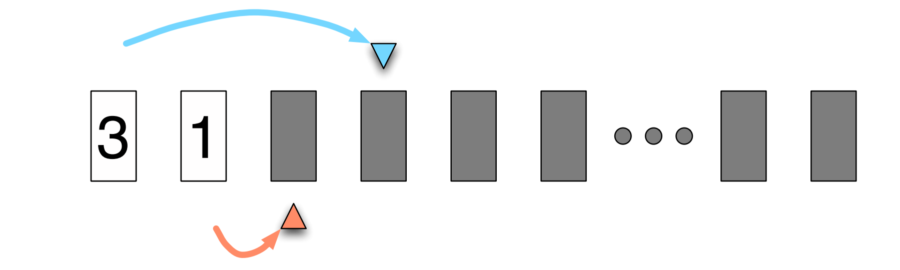

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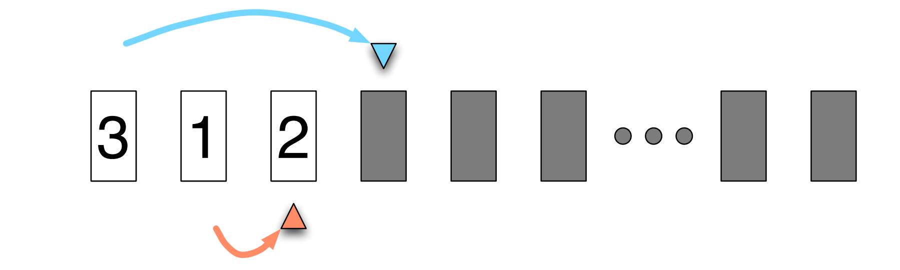

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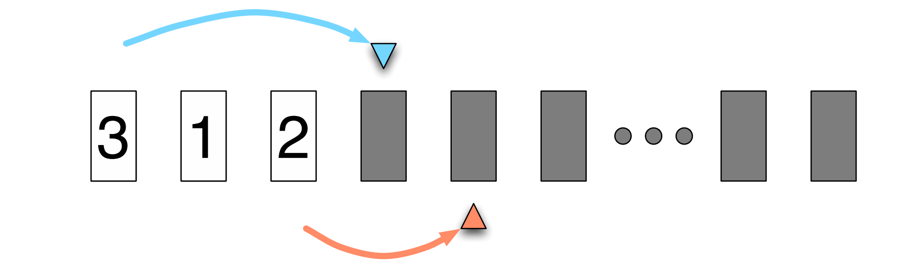

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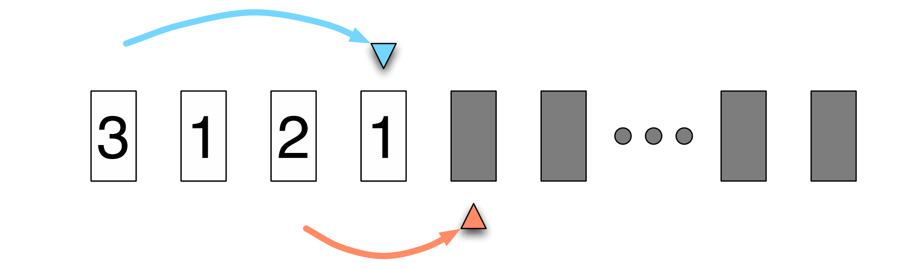

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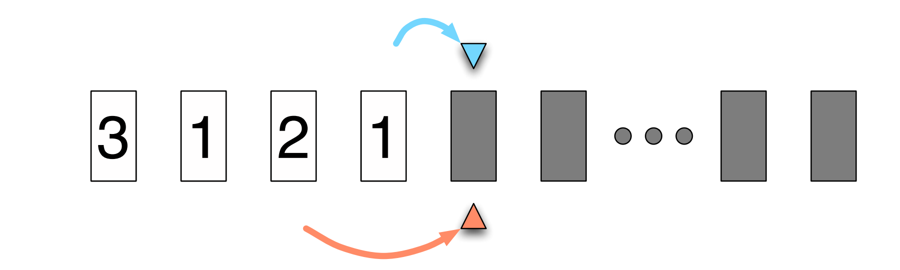

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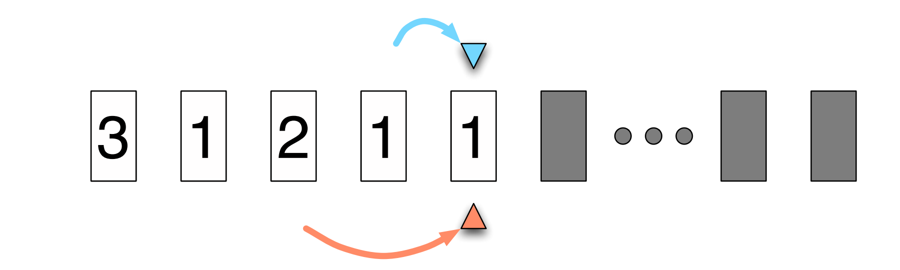

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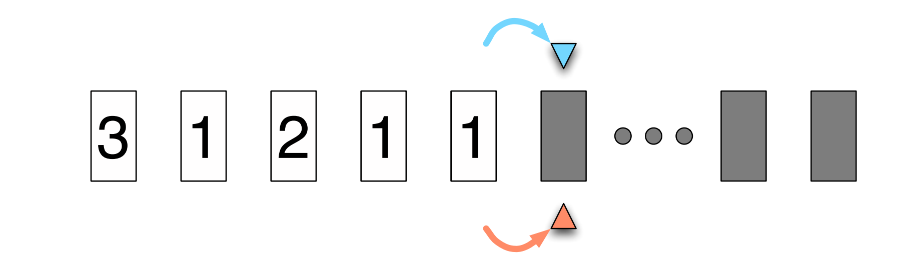

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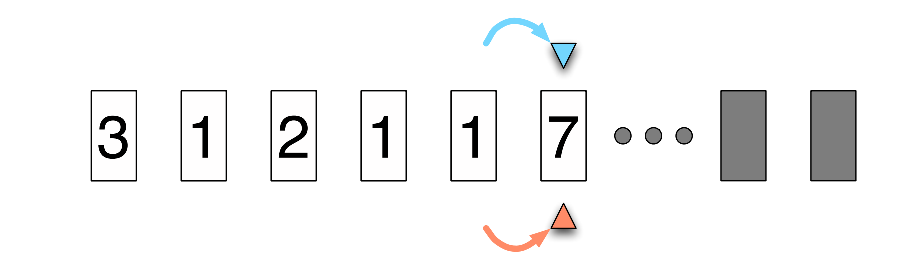

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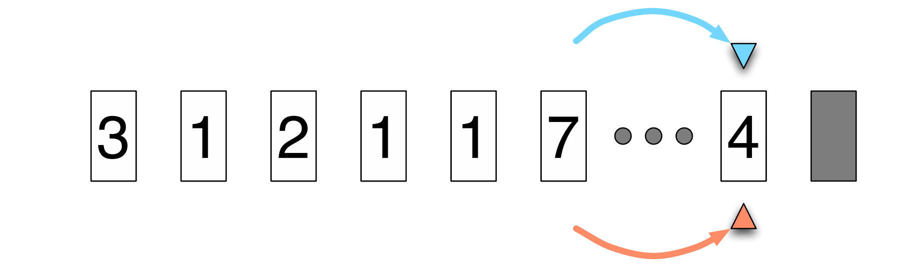

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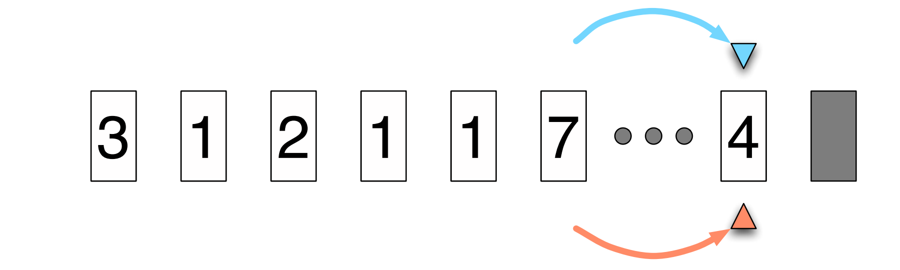


Product program: One program simulating two programs

Why is this interesting?

In general

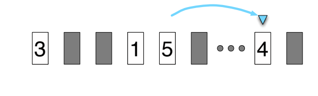
Property P of product program

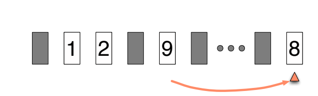
$$\Downarrow$$

Property P' of two programs

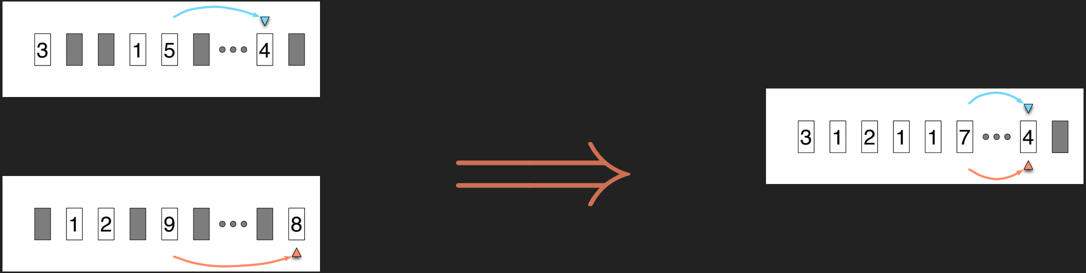
Our construction

Two simulated programs can share randomness





Distance between output distributions



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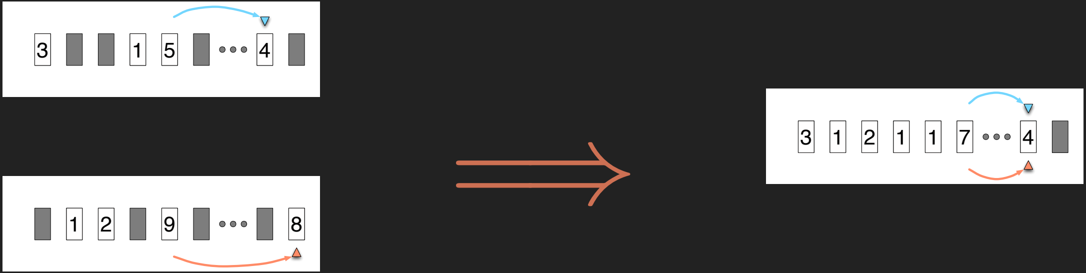
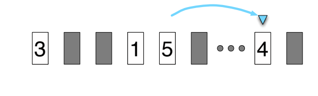
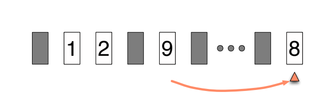
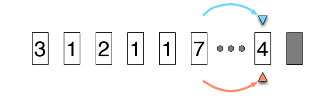


Distance between output distributions $\leq$ Probability that outputs differ





Today:



Distance between output distributions $\leq$ Probability that outputs differ

Our technical contributions

A probabilistic product construction
with shared randomness

A probabilistic program logic $\times$pRHL:
a proof-relevant version of pRHL

A crash course: Probabilistic Relational Hoare Logic [BGZ-B]

Imperative language

$$c ::= x \leftarrow e \mid c \,;\, c \mid \textsf{if } e \textsf{ then } c \textsf{ else } c \mid \textsf{while } e \textsf{ do } c$$

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Uniform sampling from finite set $[S]$

- coin flip: [heads, tails]
- random card: [1, . . . , 9]

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Uniform sampling from finite set $[S]$

- coin flip: [heads, tails]
- random card: [1, . . . , 9]

Command semantics $[\![c]\!]$

- Input: memory
- Output: distribution over memories

Judgments: similar to Hoare logic

$$\{P\}\ c\ \{Q\}$$

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Assertions: binary relation on memories

- Can refer to tagged program variables: $x\langle 1\rangle$ and $x\langle 2\rangle$
- First order formulas, non-probabilistic

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- First order formulas, non-probabilistic

> If the two inputs satisfy P, we can share the randomness on two runs of c so that the two outputs satisfy Q.

Proof rules in pRHL: mostly similar to Hoare logic

$$\textsc{Assn}\ \frac{}{\{Q\,\{e\langle 1\rangle, e\langle 2\rangle / x\langle 1\rangle, x\langle 2\rangle\}\}\ x \leftarrow e\ \{Q\}} \qquad \textsc{Rand}\ \frac{f : S \to S \text{ bijection}}{\{\forall v \in S, Q\,\{x_1\langle 1\rangle, x_2\langle 2\rangle / v, f(v)\}\}\ x \xleftarrow{\$} [S]\ \{Q\}}$$

$$\textsc{Seq}\ \frac{\{P\}\ c\ \{Q\} \qquad \{Q\}\ c'\ \{R\}}{\{P\}\ c\,;c'\ \{R\}} \qquad \textsc{Cond}\ \frac{\models P \implies e\langle 1\rangle = e\langle 2\rangle \qquad \{P \wedge e\langle 1\rangle\}\ c\ \{Q\} \qquad \{P \wedge \neg e\langle 1\rangle\}\ c'\ \{Q\}}{\{P\}\ \mathsf{if}\ e\ \mathsf{then}\ c\ \mathsf{else}\ c'\ \{Q\}} \qquad \textsc{Loop}\ \frac{\{P \wedge e\langle 1\rangle \wedge e\langle 2\rangle\}\ c\ \{P \wedge e\langle 1\rangle = e\langle 2\rangle\}}{\{P \wedge e\langle 1\rangle = e\langle 2\rangle\}\ \mathsf{while}\ e\ \mathsf{do}\ c\ \{P \wedge \neg e\langle 1\rangle \wedge \neg e\langle 2\rangle\}}$$

$$\textsc{Conseq}\ \frac{\{P\}\ c\ \{Q\} \qquad \models P' \implies P \wedge Q \implies Q'}{\{P'\}\ c\ \{Q'\}} \qquad \textsc{Case}\ \frac{\{P \wedge R\}\ c\ \{Q\} \qquad \{P \wedge \neg R\}\ c\ \{Q\}}{\{P\}\ c\ \{Q\}}$$

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$$\textsc{Loop}\ \frac{\{P \land e\langle 1\rangle \land e\langle 2\rangle\}\ c\ \{P \land e\langle 1\rangle = e\langle 2\rangle\}}{\{P \land e\langle 1\rangle = e\langle 2\rangle\}\ \mathsf{while}\ e\ \mathsf{do}\ c\ \{P \land \neg e\langle 1\rangle \land \neg e\langle 2\rangle\}}$$

$$\textsc{Conseq}\ \frac{\{P\}\ c\ \{Q\} \qquad \models P' \implies P \land Q \implies Q'}{\{P'\}\ c\ \{Q'\}}$$

$$\textsc{Case}\ \frac{\{P \land R\}\ c\ \{Q\} \qquad \{P \land \neg R\}\ c\ \{Q\}}{\{P\}\ c\ \{Q\}}$$

Proof rules in pRHL: Random sampling

$$\frac{f : S \to S \text{ bijection}}{\{\top\}\; x \xleftarrow{\$} [S]\; \{x\langle 2\rangle = f(x\langle 1\rangle)\}}$$

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Select how to share randomness

Introducing ×pRHL

Product pRHL

Idea: Product program $c^{\times}$ simulates two processes

$$\{P\}\ c\ \{Q\}$$

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$$\{P\}\ c\ \{Q\} \leadsto c^\times$$

Idea: Product program $c^{\times}$ simulates two processes

$$\{P\}\ c\ \{Q\} \rightsquigarrow c^{\times}$$

Runs in combined memory

- Two separate copies of single memory
- Duplicate program variables: $x\langle 1\rangle$ and $x\langle 2\rangle$

Idea: Product program $c^{\times}$ simulates two processes

$$\{P\}\ c\ \{Q\} \rightsquigarrow c^{\times}$$

Runs in combined memory

- Two separate copies of single memory
- Duplicate program variables: $x\langle 1\rangle$ and $x\langle 2\rangle$

Property of $c^{\times}$ $\implies$ property of two runs of c

A tour of $\times$pRHL rules: [Seq]

In pRHL:

$$\frac{\{P\}\ c\ \{Q\} \qquad \{Q\}\ c'\ \{R\}}{\{P\}\ c\ ;\ c'\ \{R\}}$$

A tour of $\times$pRHL rules: [Seq]

In $\times$pRHL:

$$\frac{\{P\}\; c\; \{Q\} \rightsquigarrow c^{\times} \qquad \{Q\}\; c'\; \{R\} \rightsquigarrow c^{\times\prime}}{\{P\}\; c\,;\, c'\; \{R\} \rightsquigarrow c^{\times}\,;\, c^{\times\prime}}$$

A tour of $\times$pRHL rules: [Seq]

In $\times$pRHL:

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Sequence product programs

A tour of $\times$pRHL proof rules: [Rand]

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A tour of $\times$pRHL proof rules: [Rand]

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Sample $x\langle 2\rangle$ depends on $x\langle 1\rangle$

A tour of $\times$pRHL rules: [Case]

In pRHL:

$$\frac{\{P \wedge Q\}\ c\ \{R\} \qquad \{P \wedge \neg Q\}\ c\ \{R\}}{\{P\}\ c\ \{R\}}$$

A tour of $\times$pRHL rules: [Case]

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A tour of $\times$pRHL rules: [Case]

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Case in proof $\rightsquigarrow$ conditional in product

See the paper for ...

Verifying rapid mixing for Markov chains

- Examples from statistical physics
- A cool card trick

Advanced proof rules

- Asynchronous loop rule

Soundness

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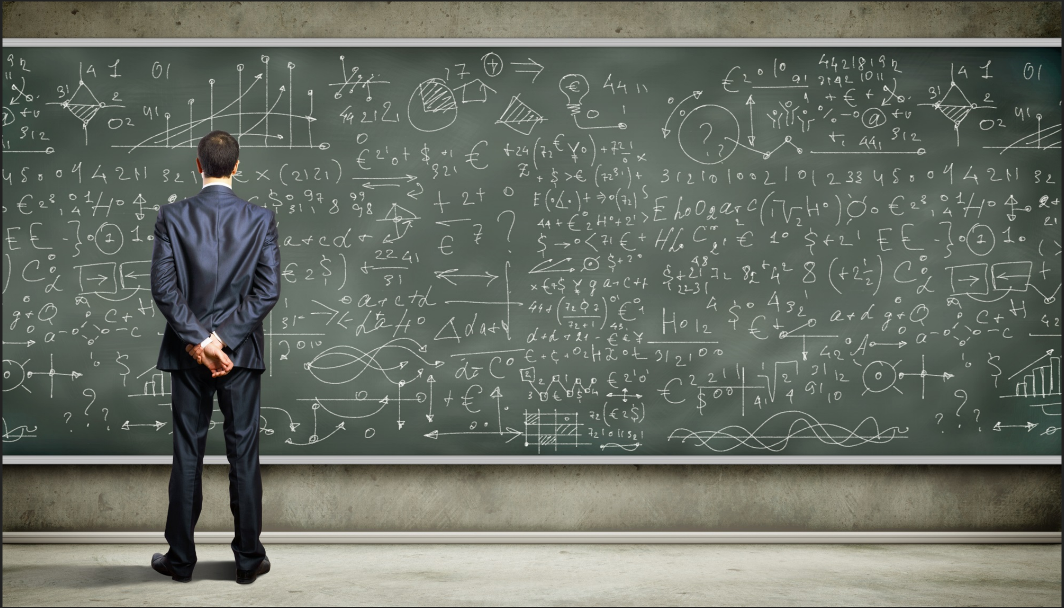

Proof by coupling

A proof technique from probability theory

- Given: two processes
- Specify: how to coordinate random samplings
- Analyze: properties of linked/coupled processes

Attractive features

- Compositional
- Reason about relation between samples, not probabilities
- Reduce properties of two programs to properties of one program

Coupling proofs $\approx$ pRHL proofs

Coupling proofs $\approx$ pRHL proofs

describe

Two coupled processes

Coupling proofs ≈ pRHL proofs

describe | encode

Two coupled processes ≈ Probabilistic product programs

Coupling proofs ≈ pRHL proofs

describe

encode

Two coupled processes ≈ Probabilistic product programs

Probabilistic product programs are the computational content of coupling proofs